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Beta Risk in the Cross-Section of Equities

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Abstract

We develop a continuous-time intertemporal CAPM model that allows for risky beta exposure, which we explicitly specify. In the model, the expected return on a stock depends on beta's co-movement with market variance and more generally with the stochastic discount factor and deviates from the standard security market line when beta risk is priced. When estimating the model on returns and options we find that allowing for beta risk helps explain the expected returns on the low and high beta stocks, which are challenging for standard factor models.

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1 Introduction

The exposure of a stock to market fluctuations is arguably its most important risk characteristic. But after more than fifty years of research the pricing of this market risk exposure is still controversial. Black, Jensen, and Scholes (1972) show that the cross-sectional relationship between estimates of beta and average stock return is too flat compared with the prediction of the classic CAPM model (Sharpe, 1964; Lintner, 1965; Mossin, 1966). An extensive subsequent literature documents pronounced time-variation in the market exposure of stock returns.[1] But, as forcefully argued by Lewellen and Nagel (2006), the empirical performance of these conditional versions of the classic CAPM is still controversial.[2]

Frazzini and Pedersen (2014) estimate betas using daily data and find that low-beta stocks offer relatively high returns on average and vice versa generating a beta anomaly. Gilbert et. al. (2014) argue that this conclusion depends greatly on the frequency of data used in beta estimation. Liu, Stambaugh and Yuan (2017) find that controlling for idiosyncratic volatility removes much of the beta anomaly. Similarly, Cederburg and O'Doherty (2016) use instrumental variables to model a conditional beta and find that it resolves the beta anomaly. Bali, Engle and Tang (2016) estimate GARCH-style beta models and argue that the predictions of the standard conditional CAPM model holds. Our contribution is to shed light on these seemingly conflicting empirical findings by developing a new asset pricing model in which beta itself is risky in a way that interacts with the market variance, and more generally with the stochastic discount factor, and where beta is priced in the cross-section of stock returns.

The key research questions we pose are: Can we use index and equity options to pin down beta dynamics and risk premiums, which are difficult to estimate from returns only? Is beta risk priced in the cross-section of stock returns? If so can it help explain the relatively flat security market line (SML) observed in practice? And finally, is there any leftover alpha to be explained? The answer turns out to be "yes" to all of these questions.

To address our research questions, we develop a bivariate extension of Heston's (1993) stochastic volatility model in which individual equity and market returns covary dynamically, while equity and index option valuation can be done in closed form. In the empirical

[1] See for example, De Bondt and Thaler (1987), Shanken (1990), Jagannathan and Wang (1996), Lettau and Ludvigson (2001), Lustig and van Nieuwerburgh (2005), Santos and Veronesi (2006), Ang, Chen, and Xing (2006), and Bollerslev, Li and Todorov (2016).

[2] See, also, Ghysels (1998), Nagel and Singleton (2011), as well as the recent surveys in Goyal (2012) and Nagel (2013).

analysis, option prices help us pin down the beta dynamics and risk premiums implied by our model. In the model, the variance-covariance matrix of index and equity returns follows a bivariate Wishart process. Our model extends the standard CAPM by allowing beta to vary stochastically. This enables us to study the cross-sectional pricing implications of beta risk, that is, the implications of co-movement of beta with aggregate volatility and the stochastic discount factor. Our model can also be viewed as providing explicit stochastic dynamics for beta in the intertemporal CAPM model.

We proceed by first specifying physical factor dynamics for the equity market, second, assuming a stochastic discount factor (SDF) allowing for a market variance risk premium, and third, deriving equity return premiums and derivatives prices from the physical dynamics and the SDF. The pricing kernel we assume allows for a market variance risk premium which in turn implies that beta risk is priced in the cross-section of equities. Our work therefore builds on the literature on the market variance risk premium including the seminal contributions by Bollerslev, Tauchen, and Zhou (2009), Carr and Wu (2009), and Driessen, Maenhout, and Vilkov (2009).

Several studies document that the relation between beta and expected equity returns in the US is flatter than predicted by the SML (see, e.g., Black, Jensen, and Scholes, 1972; and Fama and French, 1992). In Frazzini and Pedersen (2014), leverage-constrained investors tilt their portfolios toward high-beta assets bidding up their prices relative to the low-beta stocks that require financial leverage. Our model provides an alternative channel based on systematic beta risk for explaining the weak empirical relationship between estimated beta and average stock returns.

We argue that the equity return premium consists partly of a premium for beta risk. Our model predicts that when beta is low it co-moves more strongly with the SDF and market variance than when it is high. Thus, low betas tend to increase in bad times which results in a form of “wrong-way” beta risk. To compensate low beta firms for this added risk, they earn an additional premium in our model. The expected return of high-beta firms is correspondingly less than what the standard market model predicts because their betas covary negatively with the SDF and the market variance.

Our model yields analytical solutions for equity and index option prices. Existing studies show that option prices are highly informative about underlying risk and premium dynamics.[3] We therefore estimate the model by maximizing the joint return and option likelihood for a cross-section of close to 100 stocks observed over sixteen years. Our sample includes the tech

[3]See, among others, Pan (2002), Carr and Wu (2009), and Bollerslev, Todorov, and Xu (2015).

bubble and the recent financial and sovereign debt crises. This allows us to study fluctuations in betas over pronounced economic cycles. Overall, we find that the model fits the return and option data well.

There is comprehensive evidence on variance and correlation risk premiums.[4] In contrast, the study of a beta risk premium and its implications for equity expected return is largely unexplored. We document substantial cross-sectional and temporal variation in the beta risk premium. The model-implied return premium attributed to a long-short strategy exploiting return deviations from the conditional SML due to beta risk is -2.82% during our sample period. While this number is based on model estimation, we further validate the model predictions using ordinary least squares (OLS) betas for quantile portfolios of NYSE stocks. Time-variation in betas cause deviation from the SML ranging from -2.39% to 2.20% for the high and low beta portfolios, respectively. Our model thus explains more than 60% of the return deviations from the SML of high-minus-low beta strategies. Comparing ex-ante with ex-post OLS betas, we show that the ex-post beta of the high beta portfolio subsequently co-moves negatively with the SDF and market variance. In contrast, the ex-post beta of the low beta portfolios co-moves positively with the SDF and the market variance the year following the sorting.

When estimating the CAPM conditionally, the common practice is to adopt a recursive estimation method to capture fluctuations in betas using OLS.[5] A key challenge faced by this approach is to find the right balance between bias and efficiency. On the one hand, the longer the estimation window the more potential bias there is if beta is truly dynamic. On the other hand, the shorter the estimation window the greater the loss of efficiency is in estimation. Our dynamic model offers two advantages over the standard approach: First, we can use the convenient particle filter to extract latent conditional betas from a single daily return observation once the model parameters are estimated. Second, we can forecast beta across horizons using the stock's most recent spot beta and the model-implied dynamics.

We validate the ability of the model to capture beta dynamics using rolling-window OLS betas as a benchmark. Unconditional OLS and stochastic betas are close on average. Conditionally, the model betas obtain positive loadings that are highly significant when predicting the cross-section of ex-post equity returns across various horizons. Impressively, the model's betas help predict the cross-section of equity returns up to six months ahead.

[4]See, for example, Bollerslev, Tauchen, and Zhou (2009), Carr and Wu (2009), Driessen, Maenhout, and Vilkov (2009), and Todorov (2010).

[5]This is done in Jagannathan and Wang (1996), Petkova and Zhang (2005), Lewellen and Nagel (2006), and Ang, Chen, and Xing (2006), among many others.

Our paper is related to the option-implied beta literature. Chang, Christoffersen, Jacobs, and Vainberg (2011), and Buss and Vilkov (2012) exploit the information in equity and index option prices to construct model-free estimates of beta. They conclude that option-implied betas help reduce CAPM anomalies. Christoffersen, Fournier and Jacobs (2017) obtain option-based estimates of beta from a parametric model with constant beta. We complement these studies by explicitly modelling the dynamics of beta required to study beta risk and its pricing implications.

Gouriéroux and Sufana (2006) and Gouriéroux (2006) apply Wishart processes for the pricing of credit risk and the modeling of interest rates. Buraschi, Porchia, and Trojani (2010) solve an intertemporal portfolio allocation where the dependence across countries and asset classes is captured by a Wishart dynamic. Gruber, Tebaldi, and Trojani (2015) develop an index model with time-varying Wishart jump intensity. To the best of our knowledge, our study is the first to use Wishart processes to capture the joint dynamic of market index and individual equity returns.[6]

Our work is also related to the empirical literature that study correlation dynamics. Engle (2002) develops a GARCH-style dynamic correlation model. Bali (2008) uses a bivariate GARCH model to measure the covariance of large portfolios with the market index. Engle and Kelly (2012) study a time-varying equicorrelation model in which the correlation of various pairs of stocks is equal in the cross-section. Engle (2016), Bali, Engle, and Tan (2016), and Bali and Zhou (2016) develop GARCH-style beta models. Campbell, Giglio, Polk, and Turley (2016) study the pricing of volatility risk in stock returns using an intertemporal CAPM with stochastic volatility but constant correlation. Patton and Verardo (2012) investigate whether stock betas vary with earning announcements using daily betas estimated from intraday prices. We complement these papers by showing that, when beta is stochastic, the covariance of beta with the SDF and the market variance has important pricing implications in the cross-section.

2 Building a Market Model with Stochastic Beta

In this section we first define the overall modeling framework allowing for stochastic beta in a market index model. We then provide details on the specification of the variance and covariance drift and diffusion terms. Finally, we provide a stochastic discount factor that

[6] See Bru (1991), Da Fonseca, Grasselli, and Tebaldi (2007), and Da Fonseca and Grasselli (2011), Mayerhofer (2012) for other studies on Wishart processes.

allows us to pin down risk-premiums and option prices.

2.1 The Modeling Framework

Consider a market index, I_t, model with stock price, S_t, and with physical dynamics of the form

$$\begin{bmatrix} \frac{dI_t}{I_t} \\ \frac{dS_t}{S_t} \end{bmatrix} = \begin{bmatrix} r + \mu_{I,t} \\ r + \beta_t \mu_{I,t} \end{bmatrix} dt + \sqrt{\Sigma_t} \begin{bmatrix} dZ_{I,t} \\ dZ_{S,t} \end{bmatrix}, \tag{2.1}$$

where we specify the stochastic instantaneous (spot) beta as

$$\beta_t \equiv \sigma_{SI,t}/\sigma_{I,t}^2, \tag{2.2}$$

and where $\sigma_{SI,t}$ is the spot covariance between the stock and the index, and $\sigma_{I,t}^2$ is the market index spot variance. In equation (2.1), $dZ_{I,t}$ denotes market return risk and $dZ_{S,t}$ is the idiosyncratic equity shock. We thus assume that the continuous-time conditional CAPM holds and that the equity premium, $\mu_{I,t}$, is the slope of the instantaneous SML.

The matrix square root of the conditional variance of market and equity returns, $\sqrt{\Sigma_t}$, is specified as

$$\sqrt{\Sigma_t} = \begin{bmatrix} \sigma_{I,t} & 0 \\ \sigma_{SI,t}/\sigma_{I,t} & \sqrt{\sigma_{S,t}^2 - \sigma_{SI,t}^2/\sigma_{I,t}^2} \end{bmatrix}, \tag{2.3}$$

where $\sigma_{S,t}^2$ is the spot variance of the stock. Note that from (2.3) we get the spot variance matrix

$$\Sigma_t \equiv \begin{bmatrix} \sigma_{I,t}^2 & \sigma_{SI,t} \\ \sigma_{SI,t} & \sigma_{S,t}^2 \end{bmatrix} \tag{2.4}$$

We model the dynamics Σ_t as a bivariate Wishart process

$$d\Sigma_t = \left(K\left(\Theta - \Sigma_t\right) + \left(\Theta - \Sigma_t\right)K'\right)dt + \sqrt{\Sigma_t}dW_tQ + \left(\sqrt{\Sigma_t}dW_tQ\right)', \tag{2.5}$$

in which all components are 2×2 matrices. Note that K captures the mean-reversion speed of Σ_t toward the long-run values Θ, and Q captures the volatilities and co-volatilities of Σ_t. The long-run variance matrix Θ solves the system of equations given by $\gamma Q'Q - K\Theta - \Theta K' = 0$, where γ is a scalar to be estimated.[7]

[7] The parameter restriction $\gamma \geqslant N+1$ in a N-dimensional set-up ensures that the Wishart process admits a unique strong solution in the set of positive-definite matrices. In our bivariate setting it implies $\gamma \geqslant 3$.

Four independent Brownian motions in W_t drive the dynamics of the variance matrix, Σ_t in (2.5). We label them as follows

$$W_t \equiv \begin{bmatrix} W^1_{I,t} & W^2_{I,t} \\ W^1_{S,t} & W^2_{S,t} \end{bmatrix}, \tag{2.6}$$

where $W^1_{I,t}$ and $W^2_{I,t}$ capture market variance risks and $W^1_{S,t}$ and $W^2_{S,t}$ denote firm specific variance risks, respectively.

We account for the leverage effect (Black, 1976; and Christie, 1982) by linking the Brownian motions in returns and variances according to

$$\begin{aligned} dZ_{I,t} &= \sqrt{1-\rho^2}dB_{I,t} + \rho dW^1_{I,t} \\ dZ_{S,t} &= \sqrt{1-\rho^2}dB_{S,t} + \rho dW^1_{S,t}, \end{aligned} \tag{2.7}$$

where ρ is the leverage correlation parameter, and $B_{I,t}$ and $B_{S,t}$ are two independent Brownian motions. Note that our bivariate model with variance and covariance dynamics thus has a total number of 6 independent shocks.

In summary, we are following an intertemporal CAPM (ICAPM) approach but with a fully specified dynamic structure on the covariance matrix of the shocks. In our view the Wishart dynamic covariance process provides a good balance between flexibility and parameter parsimony. It enables us to pin down the term structure of risk-premiums and to price equity and index options in closed form, which is important in our empirical work below. The Wishart process allows us to build in mean-reversion in variance and covariance, a leverage effect, and non-trivial covariance dynamics which in turn generate non-trivial dynamics in β_t and a flexible term structure of beta. Importantly, we will show in Section 3.4 below how the expected return on a stock in the model deviates from the instantaneous SML when beta is priced and the investment horizon is longer than an instant.

2.2 Variance Drifts

The drift terms for the variances and covariances can be written

$$\frac{E_t\left[d\sigma^2_{I,t}\right]}{dt} = 2K_I(\theta_I - \sigma^2_{I,t}) \tag{2.8}$$

$$\frac{E_t\left[d\sigma_{SI,t}\right]}{dt} = (K_I + K_S)(\theta_{SI} - \sigma_{SI,t}) + K_{SI}(\theta_I - \sigma^2_{I,t}) \tag{2.9}$$

$$\frac{E_t\left[d\sigma^2_{S,t}\right]}{dt} = 2K_S(\theta_S - \sigma^2_{S,t}) + 2K_{SI}(\theta_{SI} - \sigma_{SI,t}), \tag{2.10}$$

where $E_t[\cdot]$ denotes the physical conditional expectations operator, and where we have defined

$$K \equiv \begin{bmatrix} K_I & 0 \\ K_{SI} & K_S \end{bmatrix}, \text{ and } \Theta \equiv \begin{bmatrix} \theta_I & \theta_{SI} \\ \theta_{SI} & \theta_S \end{bmatrix}$$

Note that we have set an off-diagonal element in K to zero to ensure that the stochastic market index variance is independent of the individual stock covariance term.

From equations (2.8), (2.9), and (2.10), we see that $2K_I$, $2K_S$, and $K_I + K_S$ capture the mean reversion speed of market variance, covariance and stock variance. The second term in equations (2.9) and (2.10) reveal an interesting property of our model. Whenever $K_{SI} \neq 0$, the market variance influences the covariance dynamic which in turn impacts the stock variance drift. Thus, fluctuations in market variance directly impact the equity covariance and indirectly the equity variance through its effect on the covariance. Empirically, we find that K_{SI} is equal to -0.27 on average. When $\sigma^2_{I,t}$ is high relative to θ_I, a negative K_{SI} implies $K_{SI}(\theta_I - \sigma^2_{I,t}) > 0$ which translates into an increase in equity covariance. A high conditional covariance relative to its long-term mean in turn increases equity variance in expectation. The model can thus generates important co-movements in equity variances and covariances.

2.3 Variance Diffusions and Leverage Effects

The model's implied market variance dynamic is closely related to the mean-reverting square-root model in Heston (1993). This is apparent from (2.8) and from the diffusion coefficient of the market index variance

$$d\sigma^2_{I,t} - E^P_t\left[d\sigma^2_{I,t}\right] = \sigma_{I,t} \cdot 2\left(Q^1_I dW^1_{I,t} + Q^2_I dW^2_{I,t}\right), \tag{2.11}$$

where we have used the square root of the variance matrix in (2.3) as well as the definition

$$Q = \begin{bmatrix} Q_I^1 & Q_S^1 \\ Q_I^2 & Q_S^2 \end{bmatrix}. \tag{2.12}$$

The key difference with Heston (1993) is that $d\sigma_{I,t}^2$ in our model in (2.11) is driven by two shocks instead of one which provides additional flexibility that is important empirically.

Despite its parsimony, the model produces important contemporaneous co-movements. The diffusion of the total stock variance is

$$\begin{aligned} d\sigma_{S,t}^2 - E_t\left[d\sigma_{S,t}^2\right] &= \beta_t \sigma_{I,t} \cdot 2\left(Q_S^1 dW_{I,t}^1 + Q_S^2 dW_{I,t}^2\right) \\ &\quad + \sqrt{\sigma_{S,t}^2 - \beta_t^2 \sigma_{I,t}^2} \cdot 2\left(Q_S^1 dW_{S,t}^1 + Q_S^2 dW_{S,t}^2\right), \end{aligned} \tag{2.13}$$

where we have used the definition of spot beta provided above, $\beta_t \equiv \sigma_{SI,t}/\sigma_{I,t}^2$.

Note that the diffusion in the stock's total variance in (2.13) follows a factor structure. On the one hand, the stock's systematic volatility $\sigma_{SI,t} = \beta_t \sigma_{I,t}$ defines the loading of $d\sigma_{S,t}^2$ on market level risks. On the other hand, the stock's idiosyncratic volatility $\sqrt{\sigma_{S,t}^2 - \beta_t^2 \sigma_{I,t}^2}$ defines the way $d\sigma_{S,t}^2$ loads on the firm-specific innovations. Similar conclusions can be drawn from the dependence of the covariance diffusion on aggregate and firm-specific shocks (see Appendix A).

From equations (2.5), (2.7), (2.12), and the dynamics of $\sigma_{I,t}^2$ and $\sigma_{S,t}^2$, we can show that the market and equity leverage effects are given by ρ and the Q matrix as follows

$$\begin{aligned} \rho_I &\equiv Corr_t\left(\tfrac{dI_t}{I_t}, d\sigma_{I,t}^2\right) = \rho \frac{Q_I^1}{\sqrt{\left(Q_I^1\right)^2 + \left(Q_I^2\right)^2}}, \\ \rho_S &\equiv Corr_t\left(\tfrac{dS_t}{S_t}, d\sigma_{S,t}^2\right) = \rho \frac{Q_S^1}{\sqrt{\left(Q_S^1\right)^2 + \left(Q_S^2\right)^2}}. \end{aligned} \tag{2.14}$$

So while the specification in (2.7) relies on a single correlation parameter, ρ, the model generates different leverage effects for the market index, ρ_I, and individual equity, ρ_S, via the parameters in the Q matrix. Finally, note that including ρ, our model has a total of 9 parameters under the physical measure plus 3 price-of-risk parameters to be defined next.

2.4 A Stochastic Discount Factor

We assume that the stochastic discount factor (SDF) is linear in the market index risks and follows the dynamics

$$\frac{d\zeta_t}{\zeta_t} = -rdt - \sigma_{I,t}\left(\lambda^{R_I} dB_{I,t} + \lambda_1^{\sigma_I} dW_{I,t}^1 + \lambda_2^{\sigma_I} dW_{I,t}^2\right), \tag{2.15}$$

where $\sigma_{I,t}\lambda^{R_I}$ is the price of market return-specific risk, $B_{I,t}$, and $\sigma_{I,t}\lambda_1^{\sigma_I}$ and $\sigma_{I,t}\lambda_2^{\sigma_I}$ are the prices of the market variance risks $W_{I,t}^1$ and $W_{I,t}^2$. Note that the stock-specific innovations $Z_{S,t}$, $W_{S,t}^1$, and $W_{S,t}^2$ are deliberately assumed not to be priced in the model. Unlike standard factor models (see, among others, Jagannathan and Wang, 1996; Lewellen and Nagel, 2006), our SDF allows for market and equity systematic variance risk premiums. This is consistent with the variance risk premium literature which provides substantial evidence that market variance risk is priced.[8]

The stochastic discount factor will enable us to pin down the equity risk premium, $\mu_{I,t}$, and to derive the risk-neutral dynamics which in turn enables us to price index and equity options. The linear form of the SDF in equation (2.15) ensures that the risk-neutral dynamics of the model will be similar to the physical dynamics in equation (2.1) above. Developing an underlying economic model that implies an SDF of the form in (2.15) presents an important topic for future research. Along these lines, Malamud and Vilkov (2017) develop an interesting overlapping generations model that implies a two-factor CAPM with myopic and non-myopic betas but they do not explicitly specify the beta dynamics.

3 Model Properties

We now explore some key properties of the model. First, we present the model under the risk-neutral measure, and, second, derive the implied instantaneous return premiums. Third, we investigate the model's implications for time-variation in beta and discuss the impact of beta's covariance with market variance and the SDF. Fourth, we develop expressions for the term structure of return risk premiums, and, fifth, we derive model-implied expected future betas.

[8]See, for example, Bollerslev, Tauchen, and Zhou (2009), Carr and Wu (2009), and Driessen, Maenhout, and Vilkov (2009).

3.1 The Risk Neutral Dynamics

The physical return dynamics in (2.1) and the stochastic discount factor in (2.15) imply that the risk-neutral dynamics (see Appendix B) used in option valuation is given by

$$\begin{bmatrix} \frac{dI_t}{I_t} \\ \frac{dS_t}{S_t} \end{bmatrix} = \begin{bmatrix} r \\ r \end{bmatrix} dt + \sqrt{\Sigma_t} \begin{bmatrix} d\tilde{Z}_{I,t} \\ d\tilde{Z}_{S,t} \end{bmatrix}. \tag{3.1}$$

Note that in our setup the spot variance-covariance matrix Σ_t is the same under the two measures which implies that the instantaneous spot beta, $\beta_t \equiv \sigma_{SI,t}/\sigma_{I,t}^2$ is identical under the two measures as well.[9] Note, however that the dynamics of the Σ_t will differ under the two measures which in turn will have interesting implication for the term-structure of risk premiums as investigated below. The risk-neutral returns shocks are defined by

$$\begin{aligned} d\tilde{Z}_{I,t} &= dZ_{I,t} + \sigma_{I,t}(\sqrt{1-\rho^2}\lambda^{R_I} + \rho\lambda_1^{\sigma_I}) \\ d\tilde{Z}_{S,t} &= dZ_{S,t} \end{aligned}$$

The dynamics for Σ_t under the risk-neutral measure is as in equation (2.5) only with the matrix of mean-reversion parameters, K, replaced by

$$\tilde{K} = K + \begin{bmatrix} \lambda_1^{\sigma_I} Q_I^1 + \lambda_2^{\sigma_I} Q_I^2 & 0 \\ \lambda_1^{\sigma_I} Q_S^1 + \lambda_2^{\sigma_I} Q_S^2 & 0 \end{bmatrix}, \tag{3.2}$$

and the matrix of long-term mean Θ replaced by $\tilde{\Theta}$ which solves the non-linear system of equations $\tilde{K}\tilde{\Theta} + \tilde{\Theta}\tilde{K}' = \gamma Q'Q$. The risk-neutral variance shocks, $d\tilde{W}_t$, are provided in Appendix B.

3.2 Instantaneous Return Premiums

Comparing (2.1) and (3.1) we see that the instantaneous return risk premium for the market index is

$$E_t\left[\frac{dI_t}{I_t}\right] - E_t^Q\left[\frac{dI_t}{I_t}\right] \equiv \mu_{I,t}dt = \Lambda^I \sigma_{I,t}^2 dt, \tag{3.3}$$

[9] Note that this again resembles the Heston (1993) model where the instantaneous variance is the same under the two measures.

where $E_t^Q[\cdot]$ is the risk-neutral expectation operator and where

$$\Lambda^I \equiv \left(\sqrt{1-\rho^2}\right)\lambda^{R_I} + \rho\lambda_1^{\sigma_I},$$

as shown in Appendix B. For the empirically relevant case $\rho < 0$ (i.e. negative leverage effect), $\lambda^{R_I} > 0$ (i.e. positive price of return risk) and $\lambda_1^{\sigma_I} < 0$ (i.e. negative price of variance risk), we have $\Lambda^I > 0$.

For the stock we have the instantaneous risk premium

$$E_t\left[\frac{dS_t}{S_t}\right] - E_t^Q\left[\frac{dS_t}{S_t}\right] \equiv \beta_t\mu_{I,t}dt = \beta_t\left(\Lambda^I\sigma_{I,t}^2\right)dt = \Lambda^I\sigma_{SI,t}dt. \tag{3.4}$$

Note that as long as the leverage correlation, ρ, is non-zero the market variance price of risk, $\lambda_1^{\sigma_I}$, will impact the instantaneous market equity price of risk, Λ^I, and thus the instantaneous risk-premium on the stock.

3.3 Beta Risk

We next provide some implications for the dynamics of beta that are intended to provide further intuition for the model.

Proposition 1 *Given (2.5), the physical dynamics of market beta,* $\beta_t \equiv \sigma_{SI,t}/\sigma_{I,t}^2$*, is such that*

$$d\beta_t = \beta_t\left(\frac{(d\sigma_{SI,t} - \Phi_{SI}dt)}{\sigma_{SI,t}} - \frac{\left(d\sigma_{I,t}^2 - \Phi_I dt\right)}{\sigma_{I,t}^2}\right), \tag{3.5}$$

where $\Phi_{SI} \equiv 2\left(Q_I^1Q_S^1 + Q_I^2Q_S^2\right)$ *and* $\Phi_I \equiv 2\left(\left(Q_I^1\right)^2 + \left(Q_I^2\right)^2\right)$.

Proof. See Appendix C. ■

We see from Proposition 1 that β_t follows a two-factor dynamic. By definition, the market beta is proportional to equity covariance with the market index and is inversely proportional to the market index variance. Accordingly, the first factor in equation (3.5), $\left(d\sigma_{SI,t} - \Phi_{IS}dt\right)/\sigma_{SI,t}$, corresponds to the relative change in covariance which positively impacts the change in beta. The second factor, $\left(d\sigma_{I,t}^2 - \Phi_I dt\right)/\sigma_{I,t}^2$, captures the relative change in the market index variance and is negatively related to $d\beta_t$. As both the covariance and market variance are mean reverting processes, beta will be mean-reverting as well.

The instantaneous beta risk premium, $IBRP$, captures beta's co-movement with the SDF. We define it by

$$IBRP_{S,t} \equiv E_t\left[d\beta_t\right] - E_t^Q\left[d\beta_t\right] = -cov_t\left(d\beta_t, \frac{d\zeta_t}{\zeta_t}\right), \tag{3.6}$$

where $cov_t(\cdot,\cdot)$ denotes the conditional physical covariance. Stocks whose beta co-moves negatively with the SDF have a positive instantaneous beta risk premium. Given equation (3.5) and the SDF dynamic, the model-implied instantaneous beta risk premium takes the form

$$IBRP_{S,t} = \left(\lambda_S^{\beta} - \lambda_I^{VRP}\beta_t\right)dt = \lambda_I^{VRP}\left(\frac{\lambda_S^{\beta}}{\lambda_I^{VRP}} - \beta_t\right)dt, \tag{3.7}$$

where $\lambda_I^{VRP} \equiv \lambda_1^{\sigma_I}Q_I^1 + \lambda_2^{\sigma_I}Q_I^2$, and $\lambda_S^{\beta} \equiv \lambda_1^{\sigma_I}Q_S^1 + \lambda_2^{\sigma_I}Q_S^2$. We now discuss the implications of priced variance risks for $IBRP_S$. We relate $IBRP_S$ to stock expected returns in the next section.

The instantaneous beta risk premium in our model is implied directly by the prices of the market variance risks. When market variance risks are not priced (i.e. $\lambda_1^{\sigma_I} = \lambda_2^{\sigma_I} = 0$) we have $\lambda_I^{VRP} = 0$ and $\lambda_S^{\beta} = 0$. In this case, the instantaneous beta risk premium in (3.7) is zero regardless of β_t.

Empirically, the market variance risk premium in the US is typically found to be large and negative.[10] In our model, the instantaneous market variance risk premium is given by

$$IVRP_{I,t} \equiv E_t\left[d\sigma_{I,t}^2\right] - E_t^Q\left[d\sigma_{I,t}^2\right] = 2\lambda_I^{VRP}\sigma_{I,t}^2 dt,$$

so that a negative $IVRP_I$ implies that $\lambda_I^{VRP} < 0$.[11] A negative $IVRP_I$ implies that $W_{I,t}^1$ or $W_{I,t}^2$ are priced and $\lambda_S^{\beta} \neq 0$, which in turn implies a non-zero $IBRP_S$ in our model.

Note that the instantaneous beta risk premium in (3.7) is linear in β_t. The sign and the magnitude of the premium depends on the size of β_t relative to $\frac{\lambda_S^{\beta}}{\lambda_I^{VRP}}$. Empirically, we find that on average $\lambda_I^{VRP} = -0.71$ consistent with the market variance risk premium literature and $\lambda_S^{\beta} = -0.425$ so that $\frac{\lambda_S^{\beta}}{\lambda_I^{VRP}}$ is positive for most stocks and equals 0.6 on average. Consequently, the instantaneous beta risk premium is negative whenever $0 < \frac{\lambda_S^{\beta}}{\lambda_I^{VRP}} - \beta_t \Leftrightarrow \beta_t < \frac{\lambda_S^{\beta}}{\lambda_I^{VRP}}$ which is the case when β_t is relatively low. Low beta stocks have a negative

[10] See, among others, Bollerslev, Tauchen, and Zhou (2009), Carr and Wu (2009), and Driessen, Maenhout, and Vilkov (2009).

[11] Note again the analogy with Heston (1993).

$IBRP_S$ on average because their betas co-move positively with the SDF. In contrast, the betas of high beta stocks instead tend to co-move negatively with the SDF which causes them to have a positive $IBRP_S$.

In the model, beta co-moves with market variance as follows

$$cov_t\left(d\beta_t, d\sigma_{I,t}^2\right) = \Phi_I\left(\frac{\Phi_{IS}}{\Phi_I} - \beta_t\right)dt, \tag{3.8}$$

where Φ_{IS} and Φ_I are as given in Proposition 1 above. Empirically, we find that $\frac{\Phi_{IS}}{\Phi_I} \approx 0.68$ on average across stocks and $\Phi_I > 0$. As a result, stocks with beta above 0.68 will display negative covariance between their beta and the market variance. As we will discuss next, this has important implication for the term-structure of expected stock returns.

3.4 The Term Structure of Return Premiums

While our theoretical model is written in continuous time, when estimating and evaluating it, we need to decide on a return frequency of interest, say monthly, and we therefore now explore the model's implications for the return premium at different horizons.

To this end the following proposition provides the expressions of the conditional risk premium for market index and individual equity returns for horizon h. For ease of notation, we define by $X_{t,h} \equiv \int_t^{t+h} E_t\left[X_s\right]ds$ and $X_{t,h}^Q \equiv \int_t^{t+h} E_t^Q\left[X_s\right]ds$ the physical and risk-neutral integrated expectations of variable X from t to $t+h$, respectively. Armed with this notation, we now present the main theoretical result.

Proposition 2 *Given (2.1), (2.5), and (2.15), the h-year integrated market return premium at time t, $RP_{t,h}^I$, is given by*

$$RP_{t,h}^I \equiv E_t\left[\int_t^{t+h}\frac{dI_u}{I_u}\right] - E_t^Q\left[\int_t^{t+h}\frac{dI_u}{I_u}\right] = \Lambda^I\sigma_{I,t,h}^2, \tag{3.9}$$

where $\Lambda^I = \left(\sqrt{1-\rho^2}\right)\lambda^{R_I} + \rho\lambda_1^{\sigma_I}$ and $\sigma_{I,t,h}^2$ is the h-year expected integrated market variance under the physical measure. The stock's h-year integrated expected return premium, $RP_{t,h}^S$, is given by

$$\begin{aligned} RP_{t,h}^S &\equiv E_t\left[\int_t^{t+h}\frac{dS_u}{S_u}\right] - E_t^Q\left[\int_t^{t+h}\frac{dS_u}{S_u}\right] = \Lambda^I\sigma_{SI,t,h} && (3.10)\\ &= RP_{t,h}^{SML} + RP_{t,h}^{BRP}, && (3.11)\end{aligned}$$

where $RP^{SML}_{t,h}$ is the return premium predicted by the conditional security market line

$$RP^{SML}_{t,h} \equiv \int_t^{t+h} \left(E_t\left[\beta_u\right] E_t\left[\frac{dI_u}{I_u}\right] - E^Q_t\left[\beta_u\right] E^Q_t\left[\frac{dI_u}{I_u}\right]\right), \tag{3.12}$$

and where $RP^{BRP}_{t,h}$ is the beta return premium

$$RP^{BRP}_{t,h} \equiv \int_t^{t+h} cov_t\left(\beta_u, \mu_{I,u}\right) du - r\left(\beta_{t,h} - \beta^Q_{t,h}\right) = \Lambda^I \int_t^{t+h} cov_t\left(\beta_u, \sigma^2_{I,u}\right) du - r\left(\beta_{t,h} - \beta^Q_{t,h}\right), \tag{3.13}$$

where $\sigma_{SI,t,h}$, $\beta_{t,h}$, and $\beta^Q_{t,h}$ are the h-year expected integrated physical covariance, and physical and risk-neutral betas, respectively.

Proof. See Appendix D. ■

The market premium is thus the product of the integrated market variance and Λ^I which reflects the representative investor's aversion to return and variance risks. Similarly, we see from (3.10) that the individual equity premium is Λ^I times the integrated covariance of the stock and the market.

Proposition 2 provides a decomposition of the equity return premium. From (3.11), we see that the equity's total return premium is composed of two parts. The first component ($RP^{SML}_{t,h}$) corresponds to the difference between expected beta times expected market return under the physical and risk-neutral measures. It is the return premium predicted for the stock by the conditional security market line. The second component ($RP^{BRP}_{t,h}$) is the covariance of beta with the market risk premium along with a term capturing the difference between the physical and risk-neutral beta multiplied by the risk-free rate. We refer to $RP^{BRP}_{t,h}$ as the beta return premium as it captures the component of the return premium induced by beta's co-movements with market variance and the SDF. Note also that we will need option-based estimates to pin down the SDF and thus to compute the empirical risk-premium decompositions.

The key dynamic in $RP^{BRP}_{t,h}$ is the covariance between beta and the market variance as defined at the end of Section 3.3. Equation (3.8) shows that the beta of high beta firms co-moves negatively with market variance and vice versa. Consequently, $\Lambda^I \left(\int_t^{t+h} cov_t\left(\beta_u, \sigma^2_{I,u}\right) du\right)$ will be negative for high beta firms and positive for low beta firms given $\Lambda^I > 0$. Note that the economic magnitude of $\Lambda^I \left(\int_t^{t+h} cov_t\left(\beta_u, \sigma^2_{I,u}\right) du\right)$ in $RP^{BRP}_{t,h}$ does not only depend on the level of beta, but also on the term-structure of the variance-covariance matrix. The second term in $RP^{BRP}_{t,h}$ captures the difference between physical and risk-neutral expected integrated betas and is typically small. Intuitively, $\beta_{t,h} - \beta^Q_{t,h}$ is related to the integral of the instantaneous beta risk premium $IBRP_{S,t}$. As noted in Section 3.3, the sign of $IBRP_{S,t}$ is driven by the way beta co-moves with the SDF. Because the beta of high beta firms co-moves negatively with the SDF, $IBRP_{S,t}$ is positive on average which implies that $\beta_{t,h} - \beta^Q_{t,h} > 0$. For low beta firms, $IBRP_{S,t}$ is negative on average because their betas co-move positively with the SDF and we have $\beta_{t,h} - \beta^Q_{t,h} < 0$. As a result, $-r\left(\beta_{t,h} - \beta^Q_{t,h}\right)$ and $\Lambda^I\left(\int_t^{t+h} cov_t\left(\beta_u, \sigma^2_{I,u}\right) du\right)$ do not cancel out each other and will take on negative values on average for high beta stocks and positive values for low beta stocks.

Note that in the limit as $h \to 0$, we have

$$\begin{aligned} RP^{SML}_{t,0} &= E_t\left[\beta_t\right] E_t\left[\tfrac{dI_t}{I_t}\right] - E^Q_t\left[\beta_t\right] E^Q_t\left[\tfrac{dI_t}{I_t}\right] \\ &= \beta_t\left(E_t\left[\tfrac{dI_t}{I_t}\right] - E^Q_t\left[\tfrac{dI_t}{I_t}\right]\right) \\ &= \beta_t\left(\Lambda^I\sigma^2_{I,t}\right)dt \\ &= \Lambda^I\sigma_{SI,t}dt, \end{aligned}$$

where we have used the definition of instantaneous market return premium in equation (3.3) combined with the fact that $E_t\left[\beta_t\right] = E^Q_t\left[\beta_t\right] = \beta_t$. Moreover, we have

$$RP^{BRP}_{t,0} = \Lambda^I cov_t\left(\beta_t, \sigma^2_{I,t}\right)dt - r\left(\beta_{t,0} - \beta^Q_{t,0}\right) = 0,$$

where we have used that, conditional on time t, β_t and $\sigma^2_{I,t}$ are known and do not covary and that $\beta_{t,0} = \beta^Q_{t,0} = \beta_t dt$ by absolute continuity of the two probability measures. Thus in the limit, even if beta co-moves with market variance and the SDF it has no impact on instantaneous expected returns. This result is related to the solution of optimal portfolio allocation problems. When the investment horizon is equal to the discretization step, intertemporal hedging demands are zero and the representative investor holds her mean-variance allocation. As the horizon increases, intertemporal hedging demands of the non-myopic agent kick in and the agent's holdings deviate from her mean-variance allocation. A similar mechanism is at play in our set-up. Instantaneously, the SML holds perfectly. As the horizon increases, expected co-movements of betas with the market variance and the SDF generate deviations from the conditional SML.

The results in Proposition 2 complements an extensive literature that studies the way time-variation in conditional betas impacts unconditional CAPM alphas. Lewellen and Nagel (2006) show that unconditional alphas are function of the way beta co-moves with market return premium and market volatility. Frazzini and Pedersen (2014) show that high beta stocks earn negative unconditional alphas. In their model, constrained investors tilt their portfolios toward high-beta assets bidding up their prices. Consequently, high-beta stocks require relatively low risk-adjusted returns compared with low-beta stocks, which require leverage. Our model provides an alternative explanation to this stylized fact. High beta stocks have lower expected returns than what the SML predicts because of the way their betas co-move with market variance and the SDF. More recently, Cederburg and O'Doherty (2016) provide evidence that the beta of high-minus-low beta trading strategies tends to co-move negatively with the market premium which impacts unconditional CAPM alphas. This result is broadly consistent with our model which predicts that $\int_t^{t+h} cov_t\left(\beta_u, \mu_{I,u}\right) du = \Lambda^I \left(\int_t^{t+h} cov_t\left(\beta_u, \sigma^2_{I,u}\right) du\right)$ is negative for high beta stocks and positive for low beta stocks on average.

Other evidence in the literature suggests that beta co-movements with market return help explain the cross-section of stock conditional expected returns. Petkova and Zhang (2005) show that time variation in betas helps explain the value-premium puzzle. Ang, Chen, and Xing (2006) document that stocks which co-move more with the market index when market index returns are low (i.e. higher downside betas) have a positive risk-adjusted alpha. Our model identifies a potential channel for this result. The betas of low beta stocks co-move more negatively with market returns and positively with the SDF. Consequently, low beta stocks earn higher expected returns that what the conditional SML predicts since $cov_t\left(d\beta_t, \frac{d\zeta_t}{\zeta_t}\right) > 0 \Rightarrow \beta_{t,h} - \beta^Q_{t,h} < 0 \Rightarrow -r\left(\beta_{t,h} - \beta^Q_{t,h}\right) > 0$ and thus $RP^S_{t,h} > RP^{SML}_{t,h}$ all things being equal.

3.5 The Term Structure of Beta

Below, we want to empirically validate the model partly based on its ability to forecast ex-post realized beta. To this end we need to derive model-based expected future betas.

Proposition 3 *Conditional on time t, the h-year ahead expected integrated variance-covariance matrix under the physical measure is*

$$\Sigma_{t,h} = E_t\left[\int_t^{t+h} \Sigma_u du\right] = \int_t^{t+h} \left(e^{-K(u-t)}\Sigma_t e^{-K'(u-t)} + \Gamma_{t,u}\right) du, \tag{3.14}$$

where the expression for $\Gamma_{t,u}$, which is a function of γ, Q, and K, is given in Appendix E. The h-year ahead expected integrated beta under the physical measure is at first-order equal *to*

$$\beta_{t,h} = E_t\left[\int_t^{t+h} \beta_u du\right] \approx \frac{\sigma_{SI,t,h}}{\theta_I} - \frac{\left(\sigma^2_{I,t,h} - \theta_I h\right)\theta_{SI}}{\left(\theta_I\right)^2}, \tag{3.15}$$

where $\sigma^2_{I,t,h}$ and $\sigma_{SI,t,h}$ denote the h-year ahead expected integrated market variance and covariance under the physical measure, respectively. They are given by $\sigma^2_{I,t,h} = \Sigma^{(1,1)}_{t,h}$ and $\sigma_{SI,t,h} = \Sigma^{(2,1)}_{t,h}$ where $\Sigma^{(i,j)}_{t,h}$ corresponds to the element on the i^{th} row, j^{th} column of $\Sigma_{t,h}$.

Proof. See Appendix E. ■

Based on these results, the annualized h-year ahead integrated variance-covariance matrix and beta can be defined by

$$\Sigma^{ann.}_{t,h} = \frac{1}{h}\Sigma_{t,h} \text{ and } \beta^{ann.}_{t,h} = \frac{1}{h}\beta_{t,h}, \tag{3.16}$$

respectively. Using (3.15) and (3.16), we can obtain the model's forecast of future realized beta. By construction, beta is a non-linear function of the state variables, and an exact analytical expression for its conditional expectation does not exist. Equation (3.15) approximates expected integrated future beta based on a first-order Taylor expansion around the long-term variance-covariance means. While this expression is not exact, we have verified using Monte Carlo simulations that it closely approximates the true expected integrated beta.

In summary, our contribution is fourfold. First, the results in Proposition 2 suggest that beta risk has important implications even in a conditional asset pricing set-up. Second, our model is broadly consistent with empirical stylized facts and allows us to gain further insights on the differential pricing implications of beta risk for stocks with low and high betas. Third, our fully-specified dynamic model allows us to quantitatively validate the model predictions taking time-varying market and equity variances, leverage effects, and variance risk premiums into consideration. Finally, our model enables us to estimate time-varying betas jointly from return and option data.

4 Empirical Results

We first present the data and the model parameter estimates including risk premiums. We then analyze the ability of our stochastic betas to predict future realized OLS betas. Subsequently, we study whether our betas can predict future equity returns. Finally, we analyze the impact of beta risk on the cross-section of expected stock returns and compare it to the security market line.

4.1 Data and Model Estimation

Our empirical analysis relies on two main datasets. We obtain daily return data from CRSP and end-of-day implied-volatility surfaces from OptionMetrics. The sample starts on January 1, 1996 and ends on December 30, 2011. To assess model performance in the cross-section, it is important to have a sufficiently large number of stocks. To this end, we obtain all the constituents of the S&P500 index as of January 1, 1996 with option data available for the full sample. We filter out firms with major corporate event during the sample period. This ensures a reliable match between the return and option data. From the remaining stocks, we retain the ninety-three firms with the biggest market capitalization. This number represents a trade-off between analyzing a reasonably large number of firms while ensuring high-quality option data for each firm.

Table 1 presents summary statistics of the return data for the index and for each industry separately. The first two columns report the sample mean and variance of returns. The remaining columns report average, minimum and maximum of OLS beta, stocks' systematic risk (share of total return variation explained by the market index return), correlation with the market index, and firms' market capitalization in billion dollars. Note that means and variances are annualized. Industry averages are obtained by first calculating each statistic at the firm level and then taking the average across all firms in a given industry. OLS betas are obtained by regressing daily excess equity returns on daily excess S&P 500 returns over the full sample. Based on the beta estimated, we then calculate equity returns' systematic risk and correlation with market index returns.

When estimating the model it is important to combine returns with option data in order to obtain the best possible estimates of risk premiums. Appendix F contains the closed-form option pricing formula implied by our model.[12]

For each stock we estimate the model parameters $\Psi \equiv \{\gamma, K, Q, \rho, \lambda_1^{\sigma_I}, \lambda_2^{\sigma_I}\}$ using maximum likelihood while filtering the paths of the unobserved market variance $\{\sigma_{I,t}^2\}$, equity variance $\{\sigma_{S,t}^2\}$, and covariance $\{\sigma_{SI,t}\}$ using the particle filter. The particle filter provides a convenient method for providing real-time estimates of the daily latent betas, $\beta_t \equiv \sigma_{SI,t}/\sigma_{I,t}^2$, for each stock conditional on the dynamic model we have estimated.

[12]Firm-level statistics for return and option data are available from the authors upon request.

Our estimation methodology is based on a weekly discretization of the model using Wednesday to Wednesday close log-returns and Wednesday close options data. The estimation procedure maximizes the sum of index and individual equity returns and options log-likelihoods. We apply this approach for each of the 93 pairs of a given stock and the market index.[13] We thus reestimate the market dynamics for each stock. If the estimated market dynamics differ greatly across stocks then that is evidence of model misspecification which we will investigate further below. Specific details on the options data as well as the construction of the likelihood and on the particle filter can be found in Appendix G. A simulation-based assessment of the performance of the particle filter in the context of our model is provided in Appendix H.

4.2 Risk Premium Estimates

In Panel A of Table 2 we report the estimated model parameters. For convenience, we report the estimates averaged by industry in this table and relegate stock level results to Table A.1 in Appendix G. Parameters with subscript S are industry averages of stock-specific parameters and parameters with subscript I are industry-averages of market-level parameters which are estimated separately for each stock. Large variation in market-level parameter estimates across industries would signal model misspecification. We note that γ, K_I, Q_I^1, and Q_I^2 are quite stable across industries. The SDF parameter $\lambda_1^{\sigma_I}$ is also stable across industries, whereas for $\lambda_2^{\sigma_I}$ Financials and Materials show somewhat different parameter values than the remaining industries.

Panel B of Table 2 reports the unconditional second moments, their respective risk premiums, and measures of model fit. In line with the stability of market parameters across industries in panel A, the long term physical variance of the market index, θ_I, is also very stable across industries with an average of 3.4% corresponding to a physical volatility of 18% per year. Similarly, the unconditional market variance risk premium, $VRP_I \equiv \theta_I - \tilde{\theta}_I$, does not vary much across sectors. On average, it is -1.4% which implies an average long term risk neutral volatility of 22 %. The model estimate of market variance risk premium compares well to what is documented in the literature (see Carr and Wu, 2009). The relative stability of market index parameters across industries suggests that our model is reasonably well specified.

[13] For alternative approaches, see for example, Renault and Touzi (1996), Gourieroux, Monfort, and Renault (1993), Pan (2002), Gagliardini, Gouriéroux, and Renault (2011), Eraker (2004), and Christoffersen, Jacobs, and Mimouni (2010).

The long term variance of equities in panel B of Table 2, θ_S, ranges from 5.2% for Utilities to 17.9% for Financials. Unconditional equity variance risk premiums, $VRP_S \equiv \theta_S - \tilde{\theta}_S$, are all negative but their magnitudes vary widely. Note that Financials display the most negative VRP_S while Consumer Staples and Utilities have variance risk premium close to zero. The third and fourth columns are informative about the dispersion of long-term covariance and beta across industries. Comparing column four of Panel B in Table 2 with column three in Table 1, we see that the unconditional betas, $\beta_S \equiv \theta_{SI}/\theta_I$, are similar in magnitude to the OLS sample betas. Little is known about covariance and beta risk premiums. In that regard, Panel B is informative about the size of these (unconditional) premiums and their distribution across industries. The unconditional covariance risk premium, which we can define by $CRP_{SI} \equiv \theta_{SI} - \tilde{\theta}_{SI}$ is negative for all industries and vary from -0.8% for Consumer Staples to -2.9% for Financials. Interestingly, the unconditional beta risk premium, $BRP_S \equiv \theta_{SI}/\theta_I - \tilde{\theta}_{SI}/\tilde{\theta}_I$, has substantial cross-sectional variation and takes on positive and negative values. It is largest for Health Care firms and lowest for Financials. Finally, Panel B reveals that stocks with high unconditional variance also tend to have large unconditional betas. The industry averages of θ_S and β_S in Panel B have a correlation coefficient of 94% across industries.

In Table 3 we report the model-implied market and equity leverage effects and the parameters Φ_{IS}, Φ_I, λ_I^{VRP}, and λ_S^{β} that determine the dynamic of beta and the dependence of beta on the SDF and market variance. Given equation (2.14) and the estimated parameters we can compute the leverage effects implied by the model. The model-implied equity leverage effect, ρ_S is -0.51 on average across industries and much lower (in absolute value) than the average market index leverage effect (ρ_I) of -0.81. Because the leverage effects drives the skewness of the return distribution, this result is consistent with Bakshi, Kapadia, and Madan (2003) who document that empirically the market index skewness is more negative than individual equity skewness on average.

Both Φ_I and λ_I^{VRP} are related to the market variance dynamics and their industry averages are stable across industries as they should be if our model is well specified. More importantly, the stock-specific ratios $\lambda_S^{\beta}/\lambda_I^{VRP}$ and Φ_{IS}/Φ_I vary widely across industries generating interesting cross-sectional differences in beta risk premiums and in beta's covariance with the market variance as discussed in Section 3.3.

Three important conclusions can be drawn so far: First, market parameters are stable across the 93 estimates as shown in Table 2 and Tables A.1 in Appendix G. Second, the model-implied leverage effects and variance risk premiums are consistent with previous estimates in the literature. Finally, the model delivers unconditional beta estimates that compare well to unconditional OLS betas on average.

4.3 A Comparison of Stochastic and OLS Betas

Recall that our estimation frequency is weekly. To obtain daily estimates of the stochastic betas, we apply the particle filter on daily market index and equity log-returns. Armed with the daily conditional variance-covariance matrices, we apply the results (3.15) and (3.16) in Proposition 3 and construct annualized one-month (i.e. 21 trading days) model forecasts of future stochastic betas, $\left(\beta_{t,\frac{21}{252}}\right) \times \frac{252}{21}$, for each firm. We denote them $\beta^{SB}_{S,t,21}$. We also construct daily estimates of OLS betas. To obtain OLS beta predictions for the $h' = 21$ day horizon of future betas on day t, we regress daily excess equity returns against the market index excess returns over the last h' days

$$R^S_u = \alpha^{OLS}_{S,t,h'} + \beta^{OLS}_{S,t,h'} \times R^I_u + \varepsilon^S_u, \text{ for } u \in \{t - h' + 1, ..., t\}. \tag{4.1}$$

Accordingly, we define the OLS beta forecast for the h'-day future realized beta on day t by the loading $\beta^{OLS}_{S,t,h'}$ of the above regression. Setting $h' = 21$, we run the regression above on every day and for each firm to obtain the time-series of one-month OLS betas over the sample period. We take the h'-day ex-post realized beta on day t to be the OLS beta for the period starting on day $t + 1$ and ending on day $t + h' + 1$, and we denote it $\beta^{OLS}_{S,t+h'+1,h'}$.

In the top panel of Figure 1, we scatter plot the sample average of the daily one-month OLS beta against the average of the one-month stochastic model betas for each firm. In the bottom panel, we scatter plot the industry averages of the one-month OLS and stochastic model betas. In both panels, the solid line corresponds to the regression fit obtained from regressing OLS betas against our stochastic model betas. The top panel uncovers the close average relationship between OLS and stochastic betas. The coefficient obtained when regressing OLS unconditional betas on average stochastic betas is 1.01 and the regression R-squared is 89%. No particular outliers are apparent. However, we do see that the model and OLS betas diverge more from each other for high beta than for low beta firms.

When averaged by industry in the bottom panel, we see that OLS and stochastic betas are again close to each other. The coefficient obtained when regressing the industry average of OLS betas on stochastic betas is 1.03 and the regression R-squared is 95%. Note that the divergence of model and OLS betas is slightly larger for Financials and IT firms, which are the most volatile industries as suggested by Tables 1 and 2. We conclude that the unconditional OLS and stochastic model betas are close on average but diverge more for high beta firms and industries that are more volatile.

In Figure 2, we plot the time-series of daily one-month OLS betas (grey) and stochastic betas (black) averaged by industries. Overall, the patterns in the two beta time-series are similar across industries. For instance, note the way OLS and stochastic betas substantially increase for IT firms during the Tech bubble, and for Financials during the recent financial crisis. This is encouraging because it demonstrates the ability of our model to adequately capture large variation in equity risks during periods of high uncertainty. Note also, that we do observe periods where OLS and stochastic betas diverge substantially. Note for example how our stochastic betas take on more extreme values than the OLS betas during the last quarter of 2008 and the first quarter of 2009. The results documented in Figure 1 and 2 suggest that OLS and stochastic model betas do share some common characteristics but also diverge substantially both in the cross-section and in the time-series dimensions.

To investigate the informational content of our stochastic betas, we regress one-month ex-post realized betas against one-month expected integrated stochastic betas controlling for one-month OLS betas. Table 4 presents the regression coefficient estimates, t-statistics, and adjusted R-squared averaged by industries. Note that the stochastic beta forecasts are statistically significant for predicting future OLS betas for all industries except Industrials. We now investigate in more detail the pricing performance of our stochastic model betas.

4.4 Return Prediction using Beta

In order to analyze the ability of our model-based betas to predict returns we first construct daily measures of one-, three-, and six-month compounded ex-post realized returns. We assume 21 trading days per month in all calculations. We denote by $R^S_{t,h'}$ the h'-day ahead compounded excess return of equity S on day t. To assess the betas' predictive performance, we run cross-sectional Fama-MacBeth predictive regressions. On each day, we regress the cross-section of future realized excess equity returns on stochastic beta (SB) and OLS betas,

$$R^S_{t+1,h'} = b^0_{t,h'} + b^{SB}_{t,h'} \times \beta^{SB}_{S,t,h'} + b^{OLS}_{t,h'} \times \beta^{OLS}_{S,t,h'} + b^{Cont.}_{t,h'} \times Control^S_{t,h'} + \varepsilon^S_{t+1,h'} \text{ for all } S, \quad (4.2)$$

where $Control^S_{t,h'}$ corresponds to a vector of lagged returns.

Table 5 presents the average of the coefficients, their t-statistics computed using the Newey-West approach, and the average of the daily regressions R-squared. We set the Newey-West autocorrelation lags to the number of trading days considered for each horizon (i.e. 5, 21, 63, and 126 lags, respectively). The R-squared obtained across horizons are high. They range from 17.6% to 22%. Comparing the coefficients obtained for stochastic betas with the ones of OLS betas reveal an interesting pattern: The coefficients obtained for stochastic betas have the expected positive sign while the coefficients obtained for OLS betas are small in magnitude, take on positive and negative values, and are less significant. The weak relation between OLS betas and stock expected returns we document is consistent with previous studies (e.g., Fama and French, 1992).

Multiplying the average of $b_{t,h'}^{SB}$ in equation (4.2) by $252/h'$ provides a model-based estimate of the market return premium and therefore serves as a consistency check. Our model estimates are 9.03%, 6.61%, 7.14%, and 6.41%, respectively, for the four different return horizons. Thus, our model delivers reasonable estimates of market expected return particularly when using return horizons beyond 5 days. Overall, the model-implied market return premiums compares well to the sample average of market return of 6.74%.

4.5 Beta Risk and the Security Market Line

While it is commonly acknowledged that beta varies over time, little is known about the impact of beta risk on expected stock returns. Equation (3.11) in Proposition 2 shows the way beta risk influences expected stock returns in our model.

To test Proposition 2, we construct model-based estimates of annualized 21-day beta return premiums for each firm as follows

$$\begin{aligned} & RP_{t,\frac{21}{252}}^{BRP} \times \frac{252}{21} \\ = & \left\{ \Lambda^I \left(\int_t^{t+21/252} cov_t\left(\beta_u, \sigma_{I,u}^2\right) du \right) - r\left(\beta_{t,\frac{21}{252}} - \beta_{t,\frac{21}{252}}^Q\right) \right\} \times \frac{252}{21} \\ = & \left\{ \Lambda^I \left(\int_t^{t+21/252} \left(E_t\left[\sigma_{SI,u}\right] - E_t\left[\beta_u\right] E_t\left[\sigma_{I,u}^2\right]\right) du \right) - r\left(\beta_{t,\frac{21}{252}} - \beta_{t,\frac{21}{252}}^Q\right) \right\} \times \frac{252}{21}. \end{aligned}$$

We set the risk-free rate, r, to its sample average of 2.3% and $\Lambda^I = 1.32$. For a given firm, we apply the results in Proposition 3 given the filtered state variables and the physical parameter K and Θ to obtain daily measure of $\beta_{t,\frac{21}{252}}$. Similarly, $\beta_{t,\frac{21}{252}}^Q$ denotes the risk-neutral one-month beta premium which we calculate using the set of risk-neutral parameters. For the first term in RP^{BRP}, we use the results in Proposition 3 to obtain estimates of the physical $E_t\left[\sigma_{SI,u}\right]$, $E_t\left[\beta_u\right]$, and $E_t\left[\sigma_{I,u}^2\right]$ which we then integrate over u. Note that we multiply

RP^{BRP} by 252/21 to annualize. We construct daily estimates of RP^{BRP} for each firm and we subsequently average those within each month.

Recall that our model predicts that the beta of firms with large conditional beta co-moves negatively with market variance and positively with market returns (i.e. negatively with the SDF) which results in a negative conditional beta return premium $RP^{BRP} < 0$. Everything else equal, a negative conditional beta return premium thus reduces the expected stock return relative to the conditional security market line. Comparing Figure 2 with Figure 3, we see that this is the case for IT firms around the Tech bubble and for Financials around the financial crisis period.

Unconditionally, the magnitude of the beta return premium, which can be positive or negative, is relatively small, but at any point in time the beta return premium can be substantial. IT firms experience a negative 10% beta return premium around the Tech bubble and a positive 7% during the financial crisis. Around the financial crisis period, Financials, and Energy and Material firms' stocks were losing about 10% from beta risk. During the Tech bubble, these industries were earning roughly a 3.5% return premium due to fluctuations in beta. During normal market conditions, beta return premiums are small in magnitude–below 1% in absolute value. These numbers correspond to the impact of beta risks on stock returns after controlling for systematic risk, that is, we measure the deviation of the predicted return from the conditional SML.

Armed with the monthly RP^{BRP} estimates, we further sort firms into quantile portfolios based on their beta return premiums each month. We then compute the equally-weighted average of the beta return premium of the low and high decile portfolios on each month to obtain $RP^{BRP}_{t,Low}$ and $RP^{BRP}_{t,High}$. High beta firms have a low beta return premium while low beta firms have a high return beta premium. We then compute the sample average of $RP^{BRP}_{t,Low} - RP^{BRP}_{t,High}$ to get a model-implied estimate of the deviation of a high minus low beta trading strategy from the conditional SML. During the 1996-2011 sample period, the average deviation from the conditional SML of this strategy is -2.82% (annualized).

We next investigate whether the model predictions hold across the entire cross-section of NYSE stocks for which we of course do not have options and so cannot directly use the dynamic stochastic beta model developed above. Effectively, this constitutes a tough out-of-sample assessment of the predictions of the model. We obtain daily stock excess return data from CRSP for all common shares traded on the NYSE and use CRSP market returns to proxy for the return on the market portfolio. In each month t, we sort stocks into quantile portfolios based on ex-ante betas $\beta^{S}_{ex-ante,t}$ where ex-ante betas are obtained from regressing daily stock excess returns against daily market excess returns over the last 252 trading days, that is $\beta^{S}_{ex-ante,t} = \beta^{OLS}_{S,t,252}$. We also compute five measures of ex-post betas for each stock. The year following sorting, we regress daily excess stock returns against daily market excess returns to obtain firms' ex-post beta and get $\beta^{S}_{ex-post,t} = \beta^{OLS}_{S,t+253,252}$. Following Ang, Chen, and Xing (2006), we compute ex-post measures of high and low market return betas, $\beta^{HighRet}_{ex-post,t}$ and $\beta^{LowRet}_{ex-post,t}$, calculated by regressing excess stock returns against excess market returns on the subset of days with market returns above and below its yearly average, respectively.[14] Arguably, $\beta^{HighRet}_{ex-post,t} - \beta^{LowRet}_{ex-post,t}$ measures the ex-post co-movements of stock beta with market returns. Note that $\beta^{HighRet}_{ex-post,t} - \beta^{LowRet}_{ex-post,t} > 0$ indicates that the beta of the stock co-moves positively with market returns in the subsequent year and vice versa. We also compute ex-post measures of high and low market variance betas, $\beta^{HighVar}_{ex-post,t}$ and $\beta^{LowVar}_{ex-post,t}$, calculated by regressing excess stock returns against market excess returns on the subset of days with market squared-returns above and below its yearly median, respectively. For robustness purposes, we compute similar measures when using the VIX index to identify high and low market variance days. Note that $\beta^{HighVar}_{ex-post,t} - \beta^{LowVar}_{ex-post,t} < 0$ indicates that the beta of the stock co-moves negatively with market variance in the year following the sorting and vice versa.

Recall that our model has three main predictions. First, stocks with relatively large beta have a negative beta return premium because their betas co-move positively with market returns (i.e. negatively with the SDF) and negatively with market variance. Second, this relation should be linear in beta. The higher the conditional beta, the higher is the co-movement of beta with market returns and the lower the co-movement of beta with market variance. Third, the expected return for large beta stocks should be lower than the SML while the expected return for low beta stocks should be higher than the SML.

Panel A of Table 6 presents the value-weighted results of sorting stocks into quintile portfolios based on ex-ante OLS betas for the 1996-2011 sample period. Comparing the first with the third column confirms that the beta of high beta stocks co-move positively with market returns as $\beta^{HighRet}_{ex-post,t} - \beta^{LowRet}_{ex-post,t} > 0$. In contrast, the beta of firms with relatively low ex-ante beta negatively co-moves with market returns ex-post. This is readily apparent as $\beta^{HighRet}_{ex-post,t} - \beta^{LowRet}_{ex-post,t} < 0$ for the low ex-ante beta portfolio. In line with the model prediction, the results are monotonically increasing/decreasing in ex-ante betas. The lower the conditional beta is the lower ex-post co-movements of beta with market returns. Comparing the first with the fourth and fifth columns provides evidence that the beta of high beta stocks co-move negatively with market variance as $\beta^{HighVar}_{ex-post,t} - \beta^{LowVar}_{ex-post,t} < 0$. In contrast, the beta of

[14] Ang, Chen, and Xing (2006) sort stocks on ex-post betas whereas we sort on ex-ante betas.

firms with relatively low ex-ante beta positively co-moves with market variance ex-post. In the sixth column, we report the one-year ex-post abnormal return which we use to measure the deviation from the SML. We see that the higher the ex-ante conditional beta, the lower the abnormal returns is over the year following the sort. The alphas range from -2.39% to 2.20%. The high minus low beta strategy earns a -4.59% abnormal return with a t-statistics of 1.69. The t-statistic is low as the sample is relatively short.

The average conditional alpha of the high minus low beta trading strategy reported in Panel A of Table 6 is comparable to -2.82%, which is the average conditional SML deviation of such a strategy implied by the model. The model helps explain more than 61% (i.e. $-2.82/-4.58$) of the abnormal performance of such a strategy. Note that the model we develop is relatively stylized and thus it is to be expected that it cannot perfectly match the historical performance of high-minus-low trading strategies.

In Panel B of Table 6 we redo the analysis from Panel A but now use data for 1950 to 2016 thus greatly extending the option sample period used above. Note that the VIX index is not available throughout the extended sample. As a result, we rely on daily squared-market returns as a proxy of market variance for that sample. Comparing Panel B with Panel A we see that the results are much stronger in the longer sample. We conclude that our model's predictions are qualitatively and quantitatively supported in the cross-section of observed equity returns.

Our model further predicts that beta should negatively co-move with firm-specific risk and that an important part of co-movement in the cross-section of betas is driven by market return and variance risks. To save space, we discuss and provide empirical evidence for these predictions in Appendix I using macro and firm-specific news data from Ravenpack.

5 Summary and Conclusions

We study the implications of beta dynamics and beta risk for the cross-section of stock returns. To this end we develop a new dynamic factor model with stochastic beta. In the model, individual equity and market returns covary dynamically and their variance-covariance matrix follows a bivariate Wishart process.

Our model can be used to filter conditional betas from daily returns and it allows for closed-form option pricing formulas. The model implies a term-structure of beta that can be used to forecast future realized betas. We develop an estimation methodology that maximizes the joint likelihood of returns and options for a large cross-section of stocks observed over a period of sixteen years.

The model makes a series of predictions. First, the model shows that part of the equity premium corresponds to compensation for risky betas. Second, it predicts that deviations from the SML is related to the co-movement of beta with the SDF and market returns. When beta is relatively low, it co-moves more positively with the stochastic discount factor (SDF) and negatively with market returns. To compensate low-beta firms for this risk, they earn an additional premium beyond the SML. Empirically we find that the model predictions hold in the cross-section of firms we study.

Several issues are left for future research. First, it may be useful to extend the model, for instance by allowing for jumps in the market price.[15] Second, combining option information with high-frequency returns when estimating the parameters in our model may lead to even better inference on beta.[16] Finally, we have focused on analyzing the implications of beta dynamics and risk for stock returns, but additionally analyzing option returns through the lens of our model would be of great interest.[17]

Appendix

This appendix collects proofs of the propositions in the main paper and additionally provides option valuation formulas for index and equity options.

A. The Physical Dynamics of $\sigma^2_{I,t}$, $\sigma_{SI,t}$, and $\sigma^2_{S,t}$

We use the dynamic of Σ_t in equation (2.5) and the form of $\sqrt{\Sigma_t}$ in (2.3) to express the dynamics of $\sigma^2_{I,t}$, $\sigma_{SI,t}$, and $\sigma^2_{S,t}$ setting the off-diagonal element in the matrix K to 0. The dynamic of $\sigma^2_{I,t}$ is

$$d\sigma^2_{I,t} = 2K_I(\theta_I - \sigma^2_{I,t})dt + 2\sigma_{I,t}\left(Q^1_I dW^1_{I,t} + Q^2_I dW^2_{I,t}\right), \tag{A.1}$$

[15] See, for example, Bates (2008), Bollerslev and Todorov (2011), Kelly, Lustig, and van Nieuwerburgh (2016).

[16] See, for example, Andersen, Fusari, and Todorov (2015), Patton and Verardo (2012), and Bollerslev, Li, and Todorov (2016).

[17] See An, Ang, Bali, and Caciki (2014) for a joint investigation of stock and option returns.

while the total individual equity variance follows

$$\begin{aligned} d\sigma^2_{S,t} &= \left(2K_S(\theta_S - \sigma^2_{S,t}) + 2K_{SI}(\theta_{SI} - \sigma_{SI,t})\right) dt \\ &\quad + 2\beta_t \sigma_{I,t} \left(Q^1_S dW^1_{I,t} + Q^2_S dW^2_{I,t}\right) \\ &\quad + 2\sqrt{\sigma^2_{S,t} - \beta^2_t \sigma^2_{I,t}} \left(Q^2_S dW^2_{S,t} + Q^1_S dW^1_{S,t}\right), \end{aligned} \tag{A.2}$$

and the covariance dynamic between the market index and the individual equity return follows

$$\begin{aligned} d\sigma_{SI,t} &= \left(K_{SI}(\theta_I - \sigma^2_{I,t}) + (K_S + K_I)(\theta_{SI} - \sigma_{SI,t})\right) dt \\ &\quad + \left(\sigma_{I,t} Q^1_S + \beta_t \sigma_{I,t} Q^1_I\right) dW^1_{I,t} + \left(\sigma_{I,t} Q^2_S + \beta_t \sigma_{I,t} Q^2_I\right) dW^2_{I,t} \\ &\quad + \sqrt{\sigma^2_{S,t} - \beta^2_t \sigma^2_{I,t}} \left(Q^1_I dW^1_{S,t} + Q^2_I dW^2_{S,t}\right). \end{aligned} \tag{A.3}$$

B. Return and Variance-Covariance Risk-Neutral Dynamics

We now derive the return and variance-covariance dynamics under the risk-neutral measure. We make use of these results in Appendix D and Appendix F.

We proceed in two steps. First, we derive the model implication for the risk-neutralization of the Brownian motions driving the dynamics of the economy. Based on the risk-neutral shocks obtained, we subsequently risk-neutralize dI_t, dS_t, and $d\Sigma_t$.

We now derive the risk-neutralization of the Brownian motions $Z_{I,t}$, $Z_{S,t}$, and W_t consistent with the SDF ζ_t. To this end, let us define $L_R \equiv \begin{bmatrix} \lambda^{R_I} \\ 0 \end{bmatrix}$ and $L_V \equiv \begin{bmatrix} \lambda^{\sigma_I}_1 & \lambda^{\sigma_I}_2 \\ 0 & 0 \end{bmatrix}$. We can re-write the SDF using the following matrix notation

$$\frac{d\zeta_t}{\zeta_t} = -r dt - Tr\left[L'_V \sqrt{\Sigma_t} dW_t\right] - L'_R \sqrt{\Sigma_t} dB_t. \tag{A.4}$$

where $B_t \equiv [B_{I,t}\ B_{S,t}]'$ and $Tr[\cdot]$ is the trace operator. By application of the multivariate Girsanov theorem, we have

$$d\begin{bmatrix} \tilde{B}_{I,t} \\ \tilde{B}_{S,t} \end{bmatrix} = d\begin{bmatrix} B_{I,t} \\ B_{S,t} \end{bmatrix} + \sqrt{\Sigma_t}' L_R dt \tag{A.5}$$

$$d\begin{bmatrix}\tilde{W}^1_{I,t} & \tilde{W}^2_{I,t}\\ \tilde{W}^1_{S,t} & \tilde{W}^2_{S,t}\end{bmatrix} = d\begin{bmatrix}W^1_{I,t} & W^2_{I,t}\\ W^1_{S,t} & W^2_{S,t}\end{bmatrix} + \sqrt{\Sigma_t}' L_V dt, \tag{A.6}$$

where the tildes denote risk-neutral Brownian motions. Note that the previous system is equivalent to

$$d\begin{bmatrix}\tilde{B}_{I,t}\\ \tilde{B}_{S,t}\end{bmatrix} = d\begin{bmatrix}B_{I,t}\\ B_{S,t}\end{bmatrix} + \sigma_{I,t}\begin{bmatrix}\lambda^{R_I}\\ 0\end{bmatrix} dt$$

$$d\begin{bmatrix}\tilde{W}^1_{I,t} & \tilde{W}^2_{I,t}\\ \tilde{W}^1_{S,t} & \tilde{W}^2_{S,t}\end{bmatrix} = d\begin{bmatrix}W^1_{I,t} & W^2_{I,t}\\ W^1_{S,t} & W^2_{S,t}\end{bmatrix} + \sigma_{I,t}\begin{bmatrix}\lambda^{\sigma_I}_1 & \lambda^{\sigma_I}_2\\ 0 & 0\end{bmatrix} dt \tag{A.7}$$

Combining these results with the leverage effect decomposition of $dZ_{I,t}$, and $d\varepsilon_{j,t}$, we can infer the risk-neutral expression for the return shock dynamics

$$\begin{aligned}
d\begin{bmatrix}\tilde{Z}_{I,t}\\ \tilde{Z}_{S,t}\end{bmatrix} &= \left(\sqrt{1-\rho^2}\right) d\begin{bmatrix}\tilde{B}_{I,t}\\ \tilde{B}_{S,t}\end{bmatrix} + \rho d\begin{bmatrix}\tilde{W}^1_{I,t} & \tilde{W}^2_{I,t}\\ \tilde{W}^1_{S,t} & \tilde{W}^2_{S,t}\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}\\
&= \left(\sqrt{1-\rho^2}\right) d\begin{bmatrix}B_{I,t}\\ B_{j,t}\end{bmatrix} + \left(\sqrt{1-\rho^2}\right)\sqrt{\Sigma_t}' L_R dt\\
&\quad +\rho\left(d\begin{bmatrix}W^1_{I,t} & W^2_{I,t}\\ W^1_{S,t} & W^2_{S,t}\end{bmatrix} + \sqrt{\Sigma_t}' L_V dt\right)\begin{bmatrix}1\\ 0\end{bmatrix}\\
&= d\begin{bmatrix}Z_{I,t}\\ Z_{S,t}\end{bmatrix} + \sqrt{\Sigma_t}'\left(\left(\sqrt{1-\rho^2}\right) L_R + \rho L_V\begin{bmatrix}1\\ 0\end{bmatrix}\right) dt
\end{aligned} \tag{A.8}$$

Combining (A.7) and (A.8) implies that

$$\begin{aligned}
d\tilde{Z}_{I,t} &= dZ_{I,t} + \sigma_{I,t}(\sqrt{1-\rho^2}\lambda^{R_I} + \rho\lambda^{\sigma_I}_1)\\
d\tilde{Z}_{S,t} &= dZ_{S,t}\\
d\tilde{W}^1_{I,t} &= dW^1_{I,t} + \sigma_{I,t}\lambda^{\sigma_I}_1 dt\\
d\tilde{W}^2_{I,t} &= dW^2_{I,t} + \sigma_{I,t}\lambda^{\sigma_I}_2 dt\\
d\tilde{W}^1_{S,t} &= dW^1_{S,t}\\
d\tilde{W}^2_{S,t} &= dW^2_{S,t}.
\end{aligned} \tag{A.9}$$

The absence of arbitrage opportunities implies that $I_t = E_t\left[\frac{\zeta_T}{\zeta_t} I_T\right]$ and $S_t = E_t\left[\frac{\zeta_T}{\zeta_t} S_T\right]$ which in turn implies that the instantaneous return premium on the market index is

$$\mu_{I,t} = (\sqrt{1-\rho^2}\lambda^{R_I} + \rho\lambda^{\sigma_I}_1)\sigma^2_{I,t} = \Lambda^I\sigma^2_{I,t}, \tag{A.10}$$

where $\Lambda^I \equiv \sqrt{1-\rho^2}\lambda^{R_I} + \rho\lambda_1^{\sigma_I}$ and that the individual equity instantaneous return premium is

$$\beta_t \mu_{I,t} = \beta_t \left(\Lambda^I \sigma_{I,t}^2\right) = \Lambda^I \sigma_{SI,t}. \tag{A.11}$$

Thus, the risk-neutral dynamic for market index and individual equity returns is given by

$$\begin{bmatrix} \frac{dI_t}{I_t} \\ \frac{dS_t}{S_t} \end{bmatrix} = \begin{bmatrix} r \\ r \end{bmatrix} dt + \sqrt{\Sigma_t} \begin{bmatrix} d\tilde{Z}_{I,t} \\ d\tilde{Z}_{S,t} \end{bmatrix}. \tag{A.12}$$

The Wishart dynamics are also inpacted by the change of measure. Using $\gamma Q^{'}Q = K\Theta + \Theta K'$, we can rewrite the physical dynamics of the Wishart process

$$\begin{aligned} d\Sigma_t &= \left(K(\Theta - \Sigma_t) + (\Theta - \Sigma_t)K'\right)dt + \sqrt{\Sigma_t}dW_tQ + \left(\sqrt{\Sigma_t}dW_tQ\right)^{'} \\ &= \left(\gamma Q^{'}Q - K\Sigma_t - \Sigma_t K^{'}\right)dt + \sqrt{\Sigma_t}dW_tQ + Q'dW_t'\sqrt{\Sigma_t}^{'}. \end{aligned} \tag{A.13}$$

Given the physical dynamics above and the risk-neutralization (A.6), that is $d\tilde{W}_t = dW_t + \sqrt{\Sigma_t}^{'} L_V$ with $L_V \equiv \begin{bmatrix} \lambda_1^{\sigma_I} & \lambda_2^{\sigma_I} \\ 0 & 0 \end{bmatrix}$, we have under the risk-neutral measure

$$\begin{aligned} d\Sigma_t &= \left(\gamma Q^{'}Q - K\Sigma_t - \Sigma_t K^{'}\right)dt + \sqrt{\Sigma_t}\left(d\tilde{W}_t - \sqrt{\Sigma_t}^{'} L_V dt\right)Q \\ &\quad + Q'\left(d\tilde{W}_t - \sqrt{\Sigma_t}^{'} L_V dt\right)^{'}\sqrt{\Sigma_t}^{'} \end{aligned}$$

$$\begin{aligned} \Leftrightarrow\ & d\Sigma_t = \left(\gamma Q^{'}Q - K\Sigma_t - \Sigma_t K^{'}\right)dt + \sqrt{\Sigma_t}d\tilde{W}_tQ - \Sigma_t L_V Q dt \\ & + Q'd\tilde{W}_t^{'}\sqrt{\Sigma_t}^{'} - Q'L_V^{'}\Sigma_t dt \end{aligned}$$

$$\Leftrightarrow d\Sigma_t = \left(\gamma Q^{'}Q - \tilde{K}\Sigma_t - \Sigma_t\tilde{K}'\right)dt + \sqrt{\Sigma_t}d\tilde{W}_tQ + Q'd\tilde{W}_t^{'}\sqrt{\Sigma_t}^{'}, \tag{A.14}$$

$$\Leftrightarrow d\Sigma_t = \left(\tilde{K}\left(\tilde{\Theta} - \Sigma_t\right) + \left(\tilde{\Theta} - \Sigma_t\right)\tilde{K}'\right)dt + \sqrt{\Sigma_t}d\tilde{W}_tQ + \left(\sqrt{\Sigma_t}d\tilde{W}_tQ\right)^{'}, \tag{A.15}$$

where

$$\tilde{K} = K + Q'L_V^{'} = K + \begin{bmatrix} \lambda_1^{\sigma_I}Q_I^1 + \lambda_2^{\sigma_I}Q_I^2 & 0 \\ \lambda_1^{\sigma_I}Q_S^1 + \lambda_2^{\sigma_I}Q_S^2 & 0 \end{bmatrix}, \quad \text{and} \quad \gamma Q^{'}Q = \tilde{K}\tilde{\Theta} + \tilde{\Theta}\tilde{K}'. \tag{A.16}$$

Together, equations (A.12), (A.15), and (A.16) define the joint risk-neutral dynamics of the market index and equity returns, and of the variance-covariance matrix.

C. The Physical Dynamics of Beta

We now derive the dynamics of equity risk under the physical measure. By definition, the stock beta satisfies $\beta_t = \sigma_{SI,t}/\sigma_{I,t}^2$. A straightforward application of Itô's lemma implies that

$$d\beta_t = \frac{1}{\sigma_{I,t}^2} d\sigma_{SI,t} - \frac{\sigma_{SI,t}}{\left(\sigma_{I,t}^2\right)^2} d\sigma_{I,t}^2 + \frac{\sigma_{SI,t}}{\left(\sigma_{I,t}^2\right)^3} cov_t\left(d\sigma_{I,t}^2, d\sigma_{I,t}^2\right) - \frac{1}{\left(\sigma_{I,t}^2\right)^2} cov_t\left(d\sigma_{I,t}^2, d\sigma_{SI,t}\right), \quad \text{(A.17)}$$

where $cov_t(\cdot,\cdot)$ denotes the instantaneous covariance operator. From (A.1) and (A.3), we see that the quadratic variations $cov_t\left(d\sigma_{I,t}^2, d\sigma_{I,t}^2\right)$ and $cov_t\left(d\sigma_{I,t}^2, d\sigma_{SI,t}\right)$ satisfy

$$\frac{cov_t\left(d\sigma_{I,t}^2, d\sigma_{I,t}^2\right)}{dt} = \sigma_{I,t}^2 \cdot \left(4\left(\left(Q_I^1\right)^2 + \left(Q_I^2\right)^2\right)\right) = \sigma_{I,t}^2 \cdot A \quad \text{(A.18)}$$

where $A \equiv 4\left(\left(Q_I^1\right)^2 + \left(Q_I^2\right)^2\right)$, and

$$\begin{aligned} \frac{cov_t\left(d\sigma_{I,t}^2, d\sigma_{SI,t}\right)}{dt} &= \sigma_{I,t}^2 \cdot 2\left(Q_I^1 Q_S^1 + Q_I^2 Q_S^2\right) + \sigma_{SI,t} \cdot 2\left(\left(Q_I^1\right)^2 + \left(Q_I^2\right)^2\right) \\ &= \sigma_{I,t}^2 \cdot B + \sigma_{SI,t} \cdot C \quad \text{(A.19)} \end{aligned}$$

where $B \equiv 2\left(Q_I^1 Q_S^1 + Q_I^2 Q_S^2\right)$, and $C \equiv 2\left(\left(Q_I^1\right)^2 + \left(Q_I^2\right)^2\right)$, respectively. We can use these results to obtain

$$\begin{aligned} d\beta_t &= \frac{1}{\sigma_{I,t}^2} d\sigma_{SI,t} - \frac{\sigma_{SI,t}}{\left(\sigma_{I,t}^2\right)^2} d\sigma_{I,t}^2 + \frac{\sigma_{SI,t}}{\left(\sigma_{I,t}^2\right)^3} \sigma_{I,t}^2 A dt - \frac{1}{\left(\sigma_{I,t}^2\right)^2}\left(\sigma_{I,t}^2 \cdot B + \sigma_{SI,t} \cdot C\right) dt \\ &= \frac{1}{\sigma_{I,t}^2} d\sigma_{SI,t} - \frac{\sigma_{SI,t}}{\left(\sigma_{I,t}^2\right)^2} d\sigma_{I,t}^2 + \frac{\sigma_{SI,t}}{\left(\sigma_{I,t}^2\right)^2} A dt - \frac{1}{\sigma_{I,t}^2} B dt - \frac{\sigma_{SI,t}}{\left(\sigma_{I,t}^2\right)^2} C dt \\ &= \frac{1}{\sigma_{I,t}^2}\left(d\sigma_{SI,t} - B dt\right) - \frac{\sigma_{SI,t}}{\left(\sigma_{I,t}^2\right)^2}\left(d\sigma_{I,t}^2 - (A - C)\, dt\right) \\ &= \beta_t \left(\frac{\left(d\sigma_{SI,t} - \Phi_{IS} dt\right)}{\sigma_{SI,t}} - \frac{\left(d\sigma_{I,t}^2 - \Phi_I dt\right)}{\sigma_{I,t}^2}\right), \quad \text{(A.20)} \end{aligned}$$

where $\Phi_{IS} \equiv 2\left(Q_I^1 Q_S^1 + Q_I^2 Q_S^2\right)$ and $\Phi_I \equiv 2\left(\left(Q_I^1\right)^2 + \left(Q_I^2\right)^2\right)$ which completes the proof.

D. Market Index and Individual Equity Return Premiums

We now derive integrated return premiums from t to $t+h$. From equations (2.1) and (3.1), we have

$$E_t\left[\int_t^{t+h}\frac{dI_u}{I_u}\right]-E_t^Q\left[\int_t^{t+h}\frac{dI_u}{I_u}\right]=E_t\left[\int_t^{t+h}\left(r+\mu_{I,u}\right)du\right]-E_t^Q\left[\int_t^{t+h}rdu\right]$$
$$=E_t\left[\int_t^{t+h}\mu_{I,u}du\right]=\Lambda^I\int_t^{t+h}E_t\left[\sigma_{I,u}^2\right]du=\Lambda^I\sigma_{I,t,h}^2, \tag{A.21}$$

where we have used the form of the market return premium $\mu_{I,t}=\Lambda^I\sigma_{I,t}^2$ with $\Lambda^I=\left(\sqrt{1-\rho^2}\right)\lambda^{R_I}+\rho\lambda_1^{\sigma_I}$ (see Appendix B) and $\sigma_{I,t,h}^2\equiv\int_t^{t+h}E_t\left[\sigma_{I,u}^2\right]du$ is the h-year expected integrated market variance under the P-measure.

Comparing (2.1) and (3.1), we see that the instantaneous individual equity return premium is equal to $\beta_t\cdot\mu_{I,t}$. As a result, we have

$$E_t\left[\int_t^{t+h}\frac{dS_u}{S_u}\right]-E_t^Q\left[\int_t^{t+h}\frac{dS_u}{S_u}\right]=E_t\left[\int_t^{t+h}\left(r+\beta_u\mu_{I,u}\right)du\right]-E_t^Q\left[\int_t^{t+h}rdu\right]$$
$$=E_t\left[\int_t^{t+h}\beta_u\left(\Lambda^I\sigma_{I,u}^2\right)du\right]=\Lambda^I\int_t^{t+h}E_t\left[\sigma_{SI,u}\right]du=\Lambda^I\sigma_{SI,t,h}. \tag{A.22}$$

where $\sigma_{SI,t,h}\equiv\int_t^{t+h}E_t\left[\sigma_{SI,u}\right]du$ is the h-year expected integrated covariance under the P-measure.

The return dynamic for the individual equity in (2.1) given the form of $\sqrt{\Sigma_t}$ in (2.3) and the risk-neutralization (A.9) satisfies

$$\frac{dS_t}{S_t}=rdt+\beta_t\left(\frac{dI_t}{I_t}-rdt\right)+\left(\sqrt{\sigma_{S,t}^2-\beta_t^2\sigma_{I,t}^2}\right)dZ_{S,t}, \tag{A.23}$$

under the P-measure where $E_t\left[\frac{dI_t}{I_t}\right]=\left(r+\mu_{I,t}\right)dt$ and

$$\frac{dS_t}{S_t}=rdt+\beta_t\left(\frac{dI_t}{I_t}-rdt\right)+\left(\sqrt{\sigma_{S,t}^2-\beta_t^2\sigma_{I,t}^2}\right)d\tilde{Z}_{S,t}, \tag{A.24}$$

under the Q-measure where $E_t^Q\left[\frac{dI_t}{I_t}\right]=rdt$. The factor structure of individual equity return

(A.23) and (A.24) implies that

$$\begin{aligned}
& E_t\left[\int_t^{t+h}\frac{dS_u}{S_u}\right]-E_t^Q\left[\int_t^{t+h}\frac{dS_u}{S_u}\right] \\
&= E_t\left[\int_t^{t+h} rdu+\beta_u\left(\frac{dI_u}{I_u}-rdu\right)\right]-E_t^Q\left[\int_t^{t+h} rdu+\beta_u\left(\frac{dI_u}{I_u}-rdu\right)\right] \\
&= E_t\left[\int_t^{t+h}\beta_u\frac{dI_u}{I_u}\right]-E_t^Q\left[\int_t^{t+h}\beta_u\frac{dI_u}{I_u}\right]-r\int_t^{t+h}\left(E_t\left[\beta_u\right]-E_t^Q\left[\beta_u\right]\right)du \\
&= E_t\left[\int_t^{t+h}\beta_u\frac{dI_u}{I_u}\right]-E_t^Q\left[\int_t^{t+h}\beta_u\frac{dI_u}{I_u}\right]-r\left(\beta_{t,h}-\beta_{t,h}^Q\right). \qquad \text{(A.25)}
\end{aligned}$$

where $\beta_{t,h}\equiv\int_t^{t+h}E_t\left[\beta_u\right]du$ and $\beta_{t,h}^Q\equiv\int_t^{t+h}E_t^Q\left[\beta_u\right]du$ are the h-year expected integrated physical and risk-neutral betas, respectively. Note that we have used the fact that idiosyncratic risk is not priced in the model and thus $E_t\left[\int_t^{t+h}dZ_{S,t}\right]=E_t^Q\left[\int_t^{t+h}d\tilde{Z}_{S,t}\right]=0$.

We can now further develop the first term in (A.25). By Fubini's theorem, we have

$$E_t\left[\int_t^{t+h}\beta_u\frac{dI_u}{I_u}\right]-E_t^Q\left[\int_t^{t+h}\beta_u\frac{dI_u}{I_u}\right]=\int_t^{t+h}E_t\left[\beta_u\frac{dI_u}{I_u}\right]-\int_t^{t+h}E_t^Q\left[\beta_u\frac{dI_u}{I_u}\right].$$

Noting that

$$E_t\left[\beta_u\frac{dI_u}{I_u}\right]=E_t\left[\beta_u\right]E_t\left[\frac{dI_u}{I_u}\right]+cov_t\left(\beta_u,\frac{dI_u}{I_u}\right),$$

independently of the measure considered, we have

$$\begin{aligned}
& E_t\left[\int_t^{t+h}\beta_u\frac{dI_u}{I_u}\right]-E_t^Q\left[\int_t^{t+h}\beta_u\frac{dI_u}{I_u}\right] \qquad \text{(A.26)} \\
= & \int_t^{t+h}\left(E_t\left[\beta_u\right]E_t\left[\frac{dI_u}{I_u}\right]-E_t^Q\left[\beta_u\right]E_t^Q\left[\frac{dI_u}{I_u}\right]\right)+\int_t^{t+h}cov_t\left(\beta_u,\frac{dI_u}{I_u}\right)-cov_t^Q\left(\beta_u,\frac{dI_u}{I_u}\right).
\end{aligned}$$

We now show that

$$\begin{aligned}
\int_t^{t+h}cov_t\left(\beta_u,\frac{dI_u}{I_u}\right)-cov_t^Q\left(\beta_u,\frac{dI_u}{I_u}\right) &= \int_t^{t+h}cov_t\left(\beta_u,\mu_{I,u}\right)du \qquad \text{(A.27)} \\
&= \Lambda^I\int_t^{t+h}cov_t\left(\beta_u,\sigma_{I,u}^2\right)du \qquad \text{(A.28)}
\end{aligned}$$

First, consider $cov_t\left(\beta_u, \frac{dI_u}{I_u}\right)$ and $cov_t^Q\left(\beta_u, \frac{dI_u}{I_u}\right)$ which satisfy

$$\begin{aligned} cov_t\left(\beta_u, \frac{dI_u}{I_u}\right) &= E_t\left[\left(\beta_u - E_t\left[\beta_u\right]\right)\left(\frac{dI_u}{I_u} - E_t\left[\frac{dI_u}{I_u}\right]\right)\right] \\ cov_t^Q\left(\beta_u, \frac{dI_u}{I_u}\right) &= E_t^Q\left[\left(\beta_u - E_t^Q\left[\beta_u\right]\right)\left(\frac{dI_u}{I_u} - E_t^Q\left[\frac{dI_u}{I_u}\right]\right)\right]. \end{aligned}$$

Under the risk-neutral measure, we have $dI_u/I_u - E_t^Q\left[dI_u/I_u\right] = \sigma_{I,u} d\tilde{Z}_{I,u}$ which implies that $cov_t^Q\left(\beta_u, \frac{dI_u}{I_u}\right)$ is equal to

$$\begin{aligned} & E_t^Q\left[\left(\beta_u - E_t^Q\left[\beta_u\right]\right)\left(\sigma_{I,u} d\tilde{Z}_{I,u}\right)\right] \\ =\ & E_t^Q\left[\left(\beta_u - E_t^Q\left[\beta_u\right]\right) E_u^Q\left[\left(\sigma_{I,u} d\tilde{Z}_{I,u}\right)\right]\right] \\ =\ & 0, \end{aligned} \tag{A.29}$$

while under the physical measure, we have $dI_u/I_u - E_t\left[dI_u/I_u\right] = \mu_{I,u} - E_t\left[\mu_{I,u}\right] + \sigma_{I,u} dZ_{I,u}$ and $cov_t\left(\beta_u, \frac{dI_u}{I_u}\right)$ satisfies

$$\begin{aligned} & E_t\left[\left(\beta_u - E_t\left[\beta_u\right]\right)\left(\left(\mu_{I,u} - E_t\left[\mu_{I,u}\right]\right) + \sigma_{I,u} dZ_{I,u}\right)\right] \\ =\ & E_t\left[\left(\beta_u - E_t\left[\beta_u\right]\right)\left(\mu_{I,u} - E_t\left[\mu_{I,u}\right]\right)\right] + 0 \\ =\ & cov_t\left(\beta_u, \mu_{I,u}\right). \end{aligned} \tag{A.30}$$

Combining the results in equations (A.25), (A.26), and (A.27), we get

$$\begin{aligned} & E_t\left[\int_t^{t+h} \frac{dS_u}{S_u}\right] - E_t^Q\left[\int_t^{t+h} \frac{dS_u}{S_u}\right] \\ =\ & \int_t^{t+h}\left(E_t\left[\beta_u\right] E_t\left[\frac{dI_u}{I_u}\right] - E_t^Q\left[\beta_u\right] E_t^Q\left[\frac{dI_u}{I_u}\right]\right) + \int_t^{t+h} cov_t\left(\beta_u, \mu_{I,u}\right) du - r\left(\beta_{t,h} - \beta_{t,h}^Q\right) \\ =\ & RP_{t,h}^{SML} + RP_{t,h}^{BRP}, \end{aligned} \tag{A.31}$$

where

$$RP_{t,h}^{SML} \equiv \int_t^{t+h} \left(E_t\left[\beta_u\right] E_t\left[\frac{dI_u}{I_u}\right] - E_t^Q\left[\beta_u\right] E_t^Q\left[\frac{dI_u}{I_u}\right]\right) \tag{A.32}$$

$$RP_{t,h}^{BRP} \equiv \int_t^{t+h} cov_t\left(\beta_u, \mu_{I,u}\right) du - r\left(\beta_{t,h} - \beta_{t,h}^Q\right) \tag{A.33}$$

$$= \Lambda^I\left(\int_t^{t+h} cov_t\left(\beta_u, \sigma_{I,u}^2\right) du\right) - r\left(\beta_{t,h} - \beta_{t,h}^Q\right), \tag{A.34}$$

where we have used the definition $\mu_{I,u} = \Lambda^I \sigma_{I,u}^2$ to obtain the last equality which completes the proof.

E. Term Structure of Risks

We start by deriving the model's prediction for the expected integrated variance-covariance matrix. We then apply this result to find the expression for expected integrated beta.

An application of Itô's Lemma to $e^{Kt}\Sigma_t e^{tK'}$ where Σ_t follows

$$d\Sigma_t = \left(\gamma Q^{'}Q - K\Sigma_t - \Sigma_t K^{'}\right) dt + \sqrt{\Sigma_t} dW_t Q + Q' dW_t' \sqrt{\Sigma_t}^{'},$$

with $\gamma Q^{'}Q = K\Theta + \Theta K'$ implies

$$\begin{aligned} d\left(e^{Kt}\Sigma_t e^{tK'}\right) &= \left(e^{Kt}K\Sigma_t e^{tK'} + e^{Kt}\Sigma_t K' e^{tK'}\right) dt + e^{Kt} d\Sigma_t e^{tK'} \\ &= \gamma e^{Kt} Q^{'} Q e^{tK'} dt + e^{Kt}\left(\sqrt{\Sigma_t} dW_t Q + Q' dW_t' \sqrt{\Sigma_t}^{'}\right) e^{tK'}. \end{aligned} \tag{A.35}$$

Integrating both sides of the previous equation from t to u with $u > t$ gives

$$e^{Ku}\Sigma_u e^{uK'} - e^{Kt}\Sigma_t e^{tK'} = \gamma \int_t^u e^{Kv} Q^{'} Q e^{vK'} dv + \int_t^u e^{Kv}\left(\sqrt{\Sigma_v} dW_v Q + Q' dW_v' \sqrt{\Sigma_v}\right) e^{vK'}, \tag{A.36}$$

which implies that

$$E_t\left[e^{Ku}\Sigma_u e^{uK'}\right] = e^{Kt}\Sigma_t e^{tK'} + \gamma \int_t^u e^{Kv} Q^{'} Q e^{vK'} dv. \tag{A.37}$$

Multiplying by e^{-Ku} from the left and $e^{-uK'}$ from the right, we get

$$E_t\left[\Sigma_u\right] = e^{-Ku}\left(e^{Kt}\Sigma_t e^{tK'} + \gamma\int_t^u e^{Kv}Q'Qe^{vK'}dv\right)e^{-uK'} = e^{-K(u-t)}\Sigma_t e^{-K'(u-t)} + \Gamma_{t,u}, \tag{A.38}$$

where $\Gamma_{t,u} \equiv \gamma\int_t^u e^{-K(u-v)}Q'Qe^{-K'(u-v)}dv$. To obtain the model's h-day expected integrated variance-covariance matrix, we need to integrate the previous expression over h days. This gives

$$\Sigma_{t,h} = \int_t^{t+h} E_t\left[\Sigma_u\right]du = \int_t^{t+h}\left(e^{-K(u-t)}\Sigma_t e^{-K'(u-t)} + \Gamma_{t,u}\right)du. \tag{A.39}$$

We now derive an approximation formula for the h-day ahead model forecast of future realized beta: $\beta_{t,h} = \left(\int_t^{t+h} E_t\left[\beta_u\right]du\right)$. A first order Taylor-expansion of conditional beta around θ_{SI}/θ_I leads to

$$\beta_u = \frac{\sigma_{SI,u}}{\sigma^2_{I,u}} = \frac{\theta_{SI}}{\theta_I} + \frac{(\sigma_{SI,u} - \theta_{SI})}{\theta_I} - \frac{\left(\sigma^2_{I,u} - \theta_I\right)\theta_{SI}}{\left(\theta_I\right)^2} + O(2). \tag{A.40}$$

Taking the expectation and ignoring the errors of order greater than two, we get

$$E_t\left[\beta_u\right] \approx \frac{\theta_{SI}}{\theta_I} + \frac{\left(E_t\left[\sigma_{SI,u}\right] - \theta_{SI}\right)}{\theta_I} - \frac{\left(E_t\left[\sigma^2_{I,u}\right] - \theta_I\right)\theta_{SI}}{\left(\theta_I\right)^2} \tag{A.41}$$

$$\approx \frac{E_t\left[\sigma_{SI,u}\right]}{\theta_I} - \frac{\left(E_t\left[\sigma^2_{I,u}\right] - \theta_I\right)\theta_{SI}}{\left(\theta_I\right)^2}. \tag{A.42}$$

Integrating the previous expression from t to $t+h$, we obtain

$$\beta_{t,h} \approx \frac{\sigma_{SI,t,h}}{\theta_I} - \frac{\left(\sigma^2_{I,t,h} - \theta_I h\right)\theta_{SI}}{\left(\theta_I\right)^2} \tag{A.43}$$

where $\sigma^2_{I,t,h} = \int_t^{t+h} E_t\left[\sigma^2_{I,u}\right]du = \left(\Sigma_{t,h}\right)^{(1,1)}$ and $\sigma_{SI,t,h} = \int_t^{t+h} E_t\left[\sigma_{SI,u}\right]du = \left(\Sigma_{t,h}\right)^{(2,1)}$.

F. Index and Individual Equity Option Prices

For ease of notation, we define the integrated Brownian $\tilde{Z}_{\Sigma,t,\tau} \equiv \int_t^{t+\tau}\sqrt{\Sigma_u}d\begin{bmatrix}\tilde{Z}_{I,u}\\ \tilde{Z}_{S,u}\end{bmatrix}$ and the integrated variance-covariance matrix $\Sigma^{Int}_{t,\tau} \equiv \int_t^{t+\tau}\Sigma_u du$. Given the Q-dynamics in Appendix B for dI_t and dS_t, we can apply Itô's lemma to $\ln(P_t)$ where $P_t \equiv \left[I_t\ S_t\right]'$ and obtain after integration the following expression for log-returns

$$\ln(P_T) - \ln(P_t) = r\mathbf{1}\tau - \frac{1}{2}diag\left(\Sigma_{t,\tau}^{Int}\right) + \tilde{Z}_{\Sigma,t,\tau}, \tag{A.44}$$

where $\mathbf{1}$ is a 2×1 vector of ones and $T = t + \tau$. Therefore, the conditional characteristic function of the risk-neutral log-returns takes the form

$$\begin{aligned} \tilde{\phi}_t^{LR}(\tau, u_I, u_S) &= E_t^Q\left[\exp\left(iu'\left(\ln(P_T) - \ln(P_t)\right)\right)\right] \\ &= E_t^Q\left[\exp\left(iu'\left(r\mathbf{1}\tau - \frac{1}{2}diag\left(\Sigma_{t,\tau}^{Int}\right) + \tilde{Z}_{\Sigma,t,\tau}\right)\right)\right], \end{aligned} \tag{A.45}$$

where $u \equiv [u_I \; u_S]'$ is a 2×1 vector. Let us introduce the stochastic exponential $\xi(\cdot)$ defined by

$$\xi\left(\eta'\tilde{Z}_{\Sigma,t,\tau}\right) = \exp\left(\eta'\tilde{Z}_{\Sigma,t,\tau} - \frac{1}{2}\eta' var_t\left(\tilde{Z}_{\Sigma,t,\tau}\right)\eta\right) = \exp\left(\eta'\tilde{Z}_{\Sigma,t,\tau} - \frac{1}{2}\eta'\Sigma_{t,\tau}^{Int}\eta\right). \tag{A.46}$$

Then, we can write (A.45) as

$$\begin{aligned} \tilde{\phi}_t^{LR}(\tau, u_I, u_S) &= \exp(iu'r\mathbf{1}\tau) \cdot E_t^Q\left[\xi\left(iu'\tilde{Z}_{\Sigma,t,\tau}\right)\exp\left(\frac{iu'\Sigma_{t,\tau}^{Int}iu}{2}\right)\exp\left(-\frac{iu'diag\left(\Sigma_{t,\tau}^{Int}\right)}{2}\right)\right] \\ &= \exp(iu'r\mathbf{1}\tau) \cdot E_t^Q\left[\xi\left(iu'\tilde{Z}_{\Sigma,t,\tau}\right)\exp\left(\frac{iu'\left(\Sigma_{t,\tau}^{Int}iu - diag\left(\Sigma_{t,\tau}^{Int}\right)\right)}{2}\right)\right]. \end{aligned} \tag{A.47}$$

We can define the following change-of-measure

$$\frac{dC}{dQ}(t) \equiv \xi\left(iu'\tilde{Z}_{\Sigma,0,t}\right). \tag{A.48}$$

Combining (A.47) with the change of measure (A.48), we can write

$$\tilde{\phi}_t^{LR}(\tau, u_I, u_S) = \exp(iu'r\mathbf{1}\tau)E_t^Q\left[\frac{\frac{dC}{dQ}(T)}{\frac{dC}{dQ}(t)}\exp\left(\frac{iu'\left(\Sigma_{t,\tau}^{Int}iu - diag\left(\Sigma_{t,\tau}^{Int}\right)\right)}{2}\right)\right]$$

$$\Rightarrow \tilde{\phi}_t^{LR}(\tau, u_I, u_S) = \exp(iu'r\mathbf{1}\tau)E_t^C\left[\exp\left(\frac{iu'\left(\Sigma_{t,\tau}^{Int}iu - diag\left(\Sigma_{t,\tau}^{Int}\right)\right)}{2}\right)\right].$$

Because

$$\frac{iu'\left(\Sigma_{t,\tau}^{Int}iu - diag\left(\Sigma_{t,\tau}^{Int}\right)\right)}{2} = Tr\left[\Gamma\left(u_I, u_S\right)\Sigma_{t,\tau}^{Int}\right]$$

where

$$\Gamma\left(u_I, u_S\right) \equiv \frac{1}{2} \times \left[\begin{array}{cc} -\left(u_I\right)^2 - iu_I & -u_I u_S \\ -u_I u_S & -\left(u_S\right)^2 - iu_S \end{array}\right],$$

we have

$$\tilde{\phi}_t^{LR}\left(\tau, u_I, u_S\right) = \exp(iu' r \mathbf{1}\tau) E_t^C\left[\exp\left(Tr\left[\Gamma\left(u_I, u_S\right)\cdot\Sigma_{t,\tau}^{Int}\right]\right)\right]. \tag{A.49}$$

While $\Gamma\left(\cdot,\cdot\right)$ is function of u_I and u_S, we drop the two input arguments in the rest of the proof for ease of notation. Thus, we now refer to it simply as Γ. An extension of the multivariate Girsanov theorem to the complex plane implies that under the C-measure, we have

$$dZ_t^C = d\tilde{Z}_t - i\sqrt{\Sigma_t}' u dt,$$

where $\tilde{Z}_t \equiv \left[\tilde{Z}_{I,t}\ \tilde{Z}_{S,t}\right]'$ and

$$dW_t^C = d\tilde{W}_t - i\rho\sqrt{\Sigma_t}'\left[u\ \mathbf{0}\right]dt,$$

where $\mathbf{0}$ is a 2×1 vector of zeros. We can now infer the Wishart dynamic under the new measure given the risk-neutral dynamic (A.14)

$$d\Sigma_t = \left(\gamma Q' Q - \tilde{K}\Sigma_t - \Sigma_t\tilde{K}'\right)dt + \sqrt{\Sigma_t}d\tilde{W}_t Q + Q' d\tilde{W}_t'\sqrt{\Sigma_t}'$$

which satisfies

$$\begin{aligned}\Leftrightarrow\ d\Sigma_t = &\left(\gamma Q' Q - \tilde{K}\Sigma_t - \Sigma_t\tilde{K}'\right)dt + \sqrt{\Sigma_t}\left(d\tilde{W}_t^C + i\rho\sqrt{\Sigma_t}'\left[u\ \mathbf{0}\right]dt\right)Q \\ &+ Q'\left(d\tilde{W}_t^C + i\rho\sqrt{\Sigma_t}'\left[u\ \mathbf{0}\right]dt\right)'\sqrt{\Sigma_t}'\end{aligned}$$

$$\begin{aligned}\Leftrightarrow\ d\Sigma_t = &\left(\gamma Q' Q - \tilde{K}\Sigma_t - \Sigma_t\tilde{K}'\right)dt + \sqrt{\Sigma_t}d\tilde{W}_t^C Q + i\rho\Sigma_t\left[u\ \mathbf{0}\right]Qdt \\ &+ Q' d\tilde{W}_t^{C'}\sqrt{\Sigma_t}' + i\rho Q'\left[u\ \mathbf{0}\right]'\Sigma_t dt\end{aligned}$$

$$\Leftrightarrow d\Sigma_t = \left(\gamma Q' Q - K^C\Sigma_t - \Sigma_t K^{C\prime}\right)dt + \sqrt{\Sigma_t}d\tilde{W}_t^C Q + Q' d\tilde{W}_t^{C'}\sqrt{\Sigma_t}, \tag{A.50}$$

where

$$K^C = \tilde{K} - i\rho Q' \left[u \ \mathbf{0}\right]^{'} .$$

We can now make use of the closed-form solution for the moment generating function $E_t^C[\exp\left(Tr\left[\Gamma \cdot \Sigma_{t,\tau}^{Int}\right]\right)]$ to obtain the following expression for $\tilde{\phi}_t^{LR}(\cdot)$,

$$\tilde{\phi}_t^{LR}(\tau, u_I, u_S) = \exp\left(Tr\left[A(\tau)\cdot \Sigma_t\right] + B(\tau)\right), \tag{A.51}$$

with

$$A(\tau) = \left(a^{22}(\tau)\right)^{-1} \cdot \left(a^{21}(\tau)\right) \tag{A.52}$$

where

$$\begin{pmatrix} a^{11}(\tau) & a^{12}(\tau) \\ a^{21}(\tau) & a^{22}(\tau) \end{pmatrix} = \exp\left(\tau \begin{bmatrix} -K^C & -2Q'Q \\ \Gamma & K^{C'} \end{bmatrix}\right)$$

and

$$B(\tau) = -\frac{\gamma}{2} Tr\left[\log\left(a^{22}(\tau)\right) - \tau K^{C'} + i\tau r u' \mathbf{1}\right] \tag{A.53}$$

where

$$\begin{aligned} u &= [u_I \ u_S]' \\ K^C &= \tilde{K} - i\rho Q' \left[u \ \mathbf{0}\right]^{'} , \end{aligned}$$

and

$$\Gamma = \frac{1}{2} \times \begin{bmatrix} -\left(u_I\right)^2 - iu_I & -u_I u_S \\ -u_I u_S & -\left(u_S\right)^2 - iu_S \end{bmatrix}.$$

Given the characteristic function above the price of a call written on the market index with strike price X is

$$C_t^I(I_t, X, \tau) = I_t\left(\frac{1}{2} - \Pi_{t,\tau}^I\right), \tag{A.54}$$

and the price of a call written on the individual equity is

$$C_t^S(S_t, X, \tau) = S_t\left(\frac{1}{2} - \Pi_{t,\tau}^S\right), \tag{A.55}$$

where the risk-neutral probabilities $\Pi_{t,\tau}^{I}$ and $\Pi_{t,\tau}^{S}$ are defined by

$$\Pi_{t,\tau}^{I} = \frac{e^{-r\tau}}{2\pi}\int_{0}^{\infty}\mathrm{Re}\left[\frac{e^{-iu_I\ln X/I_t}\tilde{\phi}_t^{LR}(\tau,u_I-i,0)}{(u_I)^2-iu_I}-\frac{e^{iu_I\ln X/I_t}\tilde{\phi}_t^{LR}(\tau,-u_I-i,0)}{(u_I)^2+iu_I}\right]du_I$$

$$\Pi_{t,\tau}^{S} = \frac{e^{-r\tau}}{2\pi}\int_{0}^{\infty}\mathrm{Re}\left[\frac{e^{-iu_S\ln X/S_t}\tilde{\phi}_t^{LR}(\tau,0,u_S-i)}{(u_S)^2-iu_S}-\frac{e^{iu_S\ln X/S_t}\tilde{\phi}_t^{LR}(\tau,0,-u_S-i)}{(u_S)^2+iu_S}\right]du_S.$$

G. Return and Option Log-Likelihoods

We now describe the particle filter and optimization algorithm applied to obtain the set of model parameter $\Psi \equiv \{\gamma, K, Q, \rho, \lambda_1^{\sigma_I}, \lambda_2^{\sigma_I}\}$ and the time-series of latent variables $\{\Sigma_t\}$ for the market index and a given equity.

Applying Itô's lemma and Euler's discretization to the log-returns given equation (2.1) leads to

$$\begin{bmatrix} \log I_{t+\Delta t} \\ \log S_{t+\Delta t} \end{bmatrix} = \begin{bmatrix} \log I_t \\ \log S_t \end{bmatrix} + \begin{bmatrix} (r+\mu_{I,t}) - \frac{1}{2}\sigma_{I,t}^2 \\ (r+\beta_t\mu_{I,t}) - \frac{1}{2}\sigma_{S,t}^2 \end{bmatrix}\Delta t + \sqrt{\Sigma_t}\sqrt{\Delta t}\begin{bmatrix} Z_{I,t+\Delta t} \\ Z_{S,t+\Delta t} \end{bmatrix}, \quad \text{(A.56)}$$

where Δt is the discretization horizon. We can also discretize the Wishart dynamics using a full truncation Euler scheme of (2.5) using the fact that $K\Theta + \Theta K' = \gamma Q' Q$

$$\begin{aligned}\Sigma_{t+\Delta t} &= \Sigma_t^+ + (\gamma Q'Q - K\Sigma_t^+ + \Sigma_t^+ K')\Delta t \\ &\quad + \sqrt{\Sigma_t^+}W_{t+\Delta t}Q\sqrt{\Delta t} + \left(\sqrt{\Sigma_t^+}W_{t+\Delta t}Q_{SI}\sqrt{\Delta t}\right)' \end{aligned} \quad \text{(A.57)}$$

where $\Sigma_t^+ = U_t \times abs(L_t) \times U_t'$ for which the matrix of eigenvalues L_t satisfies $\Sigma_t = U_t \times L_t \times U_t'$. Note that in (A.57), $\sqrt{\Sigma_t^+}$ corresponds to the Cholesky decomposition of Σ_t^+ such that

$$\sqrt{\Sigma_t^+} = \begin{bmatrix} \sqrt{\left(\Sigma_t^+\right)^{(1,1)}} & 0 \\ \frac{\left(\Sigma_t^+\right)^{(2,1)}}{\sqrt{\left(\Sigma_t^+\right)^{(1,1)}}} & \frac{\sqrt{\left(\Sigma_t^+\right)^{(1,1)}\left(\Sigma_t^+\right)^{(2,2)} - \left(\left(\Sigma_t^+\right)^{(2,1)}\right)^2}}{\sqrt{\left(\Sigma_t^+\right)^{(1,1)}}} \end{bmatrix}, \quad \text{(A.58)}$$

where $\left(\Sigma_t^+\right)^{(i,j)}$ corresponds to the element on the ith row and jth column of Σ_t^+. Together, equations (A.56), (A.57), and (A.58) define the Euler discretized dynamics of index and equity returns and of the variance-covariance matrix. We make use of this in what follows.

We divide the filtering and computation of the log-likelihoods into six steps that we now discuss in details.

Step 1 - Sampling: We start the algorithm at time $t = 0$ by setting all the particles $\Sigma_0^i = \Theta \; \forall i \in \{1, 2, ..., N\}$ where Θ is the long term mean of the variance-covariance matrix Σ_t, i identifies a given particle, and N denotes the total number of particles. Note that at this stage, we set the expected returns $r + \mu_{I,t}$ and $r + \beta_t \mu_{I,t}$ to the sample average of market return, $\hat{\omega}_I$, and the average of stock equity return, $\hat{\omega}_S$, respectively.

The algorithm for filtering of the particles at time $t+1$ given time t smoothly resampled particles, Σ_t^i, is as follows. From the smoothly resampled particles, we need to simulate forward a new set of particles $\tilde{\Sigma}_{t+\Delta t}^i$. To this end, we first filter the vector of shocks to returns based on the bivariate measurement equation

$$\begin{bmatrix} Z_{I,t+\Delta t}^i \\ Z_{S,t+\Delta t}^i \end{bmatrix} = (\sqrt{\Sigma_t^i}\sqrt{\Delta t})^{-1} \cdot \left(\begin{bmatrix} \log \frac{I_{t+\Delta t}}{I_t} \\ \log \frac{S_{t+\Delta t}}{S_t} \end{bmatrix} - \begin{bmatrix} \hat{\omega}_I - \frac{1}{2}\left(\sigma_{I,t}^{i,2}\right) \\ \hat{\omega}_S - \frac{1}{2}\left(\sigma_{S,t}^{i,2}\right) \end{bmatrix} \Delta t \right). \tag{A.59}$$

Armed with the shocks $Z_{I,t+\Delta t}^i$ and $Z_{S,t+\Delta t}^i$, we can generate the matrix of variance shocks $W_{t+\Delta t}^i$ consistent with the leverage (2.7) as follows

$$W_{t+\Delta t}^i = \begin{bmatrix} W_{i,I,t+\Delta t}^1 & W_{i,I,t+\Delta t}^2 \\ W_{i,S,t+\Delta t}^1 & W_{i,S,t+\Delta t}^2 \end{bmatrix} = \begin{bmatrix} \rho Z_{I,t+\Delta t}^i + \sqrt{1-\rho^2} B_{i,I,t+\Delta t}^1 & B_{i,I,t+\Delta t}^2 \\ \rho Z_{S,t+\Delta t}^i + \sqrt{1-\rho^2} B_{i,S,t+\Delta t}^1 & B_{i,S,t+\Delta t}^2 \end{bmatrix}, \tag{A.60}$$

where ρ is the leverage parameter, and $B_{i,I,t+\Delta t}^1$, $B_{i,I,t+\Delta t}^2$, $B_{i,S,t+\Delta t}^1$, and $B_{i,S,t+\Delta t}^2$ are *i.i.d.* standard Normal random variables. Note that in this step, we impose the shocks $B_{i,I,t+\Delta t}^1$ and $B_{i,I,t+\Delta t}^2$ to be the same across all equities while $B_{i,S,t+\Delta t}^1$ and $B_{i,S,t+\Delta t}^2$ change from one equity to another. This is important to ensure the consistency of the market variance path across equities.

Given the shocks to variance $W_{t+\Delta t}^i$, we simulate the N particles forward according to

$$\begin{aligned} \tilde{\Sigma}_{t+\Delta t}^i = \; & \Sigma_t^{i,+} + (\gamma Q^{'} Q - K\Sigma_t^{i,+} + \Sigma_t^{i,+} K')\Delta t \\ & + \sqrt{\Sigma_t^{i,+}} W_{t+\Delta t}^i Q \sqrt{\Delta t} + \left(\sqrt{\Sigma_t^{i,+}} W_{t+\Delta t}^i Q \sqrt{\Delta t} \right)^{'}. \end{aligned} \tag{A.61}$$

We then ensure positive definiteness by calculating $\tilde{\Sigma}_{t+\Delta t}^{i,+}$ for each particle $\tilde{\Sigma}_{t+\Delta t}^i$.

Step 2 - Importance Sampling: With the N particles $\tilde{\Sigma}_{t+\Delta t}^{i,+}$ in hand, we can infer the probability of occurrence $\tilde{w}_{t+\Delta t}^i$ of each particle from next period log-returns (i.e. the likelihood that next period log-returns at time $t+2\Delta t$ are generated by the particle $\tilde{\Sigma}^{i,+}_{t+\Delta t}$). Conditional on $\tilde{\Sigma}^{i,+}_{t+\Delta t}$, the index and equity log-returns admit a bivariate Normal density. We make use of this to obtain $\tilde{w}^i_{t+\Delta t}$

$$\tilde{w}^i_{t+\Delta t} = \frac{1}{\sqrt{(2\pi)^2|\tilde{\Sigma}^{i,+}_{t+\Delta t}|}} \times$$
$$\exp\left(-\frac{1}{2}\begin{bmatrix} \log(\frac{I_{t+2\Delta t}}{I_{t+\Delta t}}) - \hat{\omega}_I\Delta t \\ \log(\frac{S_{t+2\Delta t}}{S_{t+\Delta t}} - \hat{\omega}_S\Delta t) \end{bmatrix}' \left(\tilde{\Sigma}^{i,+}_{t+\Delta t}\right)^{-1} \begin{bmatrix} \log(\frac{I_{t+2\Delta t}}{I_{t+\Delta t}}) - \hat{\omega}_I\Delta t \\ \log(\frac{S_{t+2\Delta t}}{S_{t+\Delta t}} - \hat{\omega}_S\Delta t) \end{bmatrix}\right). \tag{A.62}$$

Based on the $\tilde{w}^i_{t+\Delta t}$, we can obtain the probability of each particle $\breve{w}^i_{t+\Delta t}$ by normalization

$$\breve{w}^i_{t+\Delta t} = \frac{\tilde{w}^i_{t+\Delta t}}{\sum_{i=1}^N \tilde{w}^i_{t+\Delta t}}. \tag{A.63}$$

Step 3 - Resampling: Since we want to maximize the likelihood function as a function of the model parameters, we apply the algorithm of Malik and Pitt (2011) and resample smoothly $\Sigma^i_{t+\Delta t}$ from the set of particles $\tilde{\Sigma}^{i,+}_{t+\Delta t}$. This ensures that the log-likelihood function is a smooth function of parameters.

Step 4 - Filtered path and returns log-likelihood: We repeat Step 1, 2, and 3 over each week in the sample to obtain $\{\Sigma^i_t, \tilde{w}^i_t\}_{t=1,2,\ldots,T}$. Based on the particles and their probability of occurrence, the log-likelihood function is calculated as

$$\mathcal{L}^R(\Psi^P) \equiv \sum_{t=1}^T \log\left(\frac{1}{N}\sum_{i=1}^N \tilde{w}^i_t(\Psi^P)\right) \tag{A.64}$$

where $\Psi^P = \{\gamma, K, Q, \rho\}$ are the parameters driving the P-dynamics of the Wishart process. Moreover, the filtered conditional variance-covariance matrix at any time t is the average of the smoothly resampled particles:

$$\hat{\Sigma}_t = \frac{1}{N}\sum_{i=1}^N \Sigma^i_t. \tag{A.65}$$

Step 5 - Index and equity options log-likelihoods: Given $\left\{\hat{\Sigma}_t\right\}_{t=1,2,\ldots,T}$ and given the risk-neutralization (A.14), the model can be used for pricing index and equity options given a set of parameters $\Psi^Q = \{\gamma, K, Q, \rho, \lambda_1^{\sigma_I}, \lambda_2^{\sigma_I}\}$. Based on the prices obtained, the model implied volatility errors on any day t for the option contract i written on asset k are calculated as

$$\eta_{i,t}^{k} = BSIV_{i,t}^{M,k} - BSIV_{i,t}^{k}(\Psi^{Q}, \hat{\Sigma}_t), \ i = 1, \ldots, 5 \tag{A.66}$$

where $k \in \{I, S\}$, $BSIV_{i,t}^{M,k}$ denotes the quoted implied volatility, and $BSIV_{i,t}^{k}(\Psi^{Q}, \hat{\Sigma}_t)$ is the implied volatility given the model price. Note that $BSIV$ denotes the Black-Scholes implied volatility.

Recall that we observe five contracts for each asset at any point in time. To calculate the options log-likelihood for asset $k \in \{I, S\}$, we define the 1×5 vector of time t implied-volatility errors $\eta_t^k = [\eta_{1,t}^k \eta_{2,t}^k \eta_{3,t}^k \eta_{4,t}^k \eta_{5,t}^k]$ from which we construct the implied volatility moment condition matrix $\eta^k = \begin{bmatrix} \eta_1^k \\ \vdots \\ \eta_T^k \end{bmatrix}$. Based on η^k, we calculate the sample variance $\hat{V}^k = Var(\eta^k)$ of the five moment conditions and calculate the options log-likelihood for asset k as

$$\mathcal{L}_k^O = \sum_{t=1}^{T} \log \left(\frac{1}{\sqrt{(2\pi)^2 |\hat{V}^k|}} \times \exp \left(-\frac{1}{2} \eta_t^k \left(\hat{V}^k \right)^{-1} \eta_t^{k'} \right) \right). \tag{A.67}$$

The last step consists of summing the index and equity options log-likelihoods to get

$$\mathcal{L}^O(\Psi^Q) \equiv \mathcal{L}_I^O + \mathcal{L}_S^O. \tag{A.68}$$

Step 6 - Summing the log-likelihoods: The set of model parameters can now be obtained as the solution to the maximization problem

$$\{\gamma, K, Q, \rho, \lambda_1^{\sigma_I}, \lambda_2^{\sigma_I}\} = argmax(\mathcal{L}^R(\Psi^P) + \mathcal{L}^O(\Psi^Q)). \tag{A.69}$$

Due to the computational intensity when estimating the model, we apply a series of filters to reduce the size of the option sample. We restrict our focus to Wednesday-close S&P500 puts and equity calls when estimating the model.[18] To obtain a reliable representation of the term structure of volatility, we retain at-the-money options with absolute deltes of 0.5 and with 30, 91, and 182 days-to-maturity, respectively. To obtain a reliable representation of the smile, we further include 0.25 and 0.75 absolute delta options with 91 days to maturity. Five options are thus retained for the index and for each equity on any given Wednesday.

[18] S&P500 index put options are more liquid than call options. In contrast, calls are the most actively traded options for individual equities.

Parameter estimates for each stock and index pair are presented in Table A.1.

H. Simulation-Based Evidence

In order to assess the performance of the particle filter, we use our model to simulate $10,000$ paths of daily excess returns over a ten-year period for the market index, a low beta firm, and a high beta firm relying on the discretized dynamics in Appendix G. We set the market index parameters to the cross-sectional average of the estimates obtained in our empirical analysis. Specifically, we set $\gamma = 4.91$, $K_I = 2.38$, $Q_I^1 = 0.175$, $Q_I^2 = -0.0256$, $\rho = -0.835$, $\lambda_1^{\sigma_I} = 3.69$, and $\lambda_2^{\sigma_I} = -0.996$. These market parameters are used for both firms. The remaining parameters are then defined to generate an unconditional beta of 0.60 (i.e. low beta firm) or 1.60 (i.e. high beta firm). To this end, for the low-beta firm, the K and Q matrices are set to

$$K_{low} = \begin{bmatrix} 2.38 & 0 \\ -0.331 & 1.67 \end{bmatrix} \quad Q_{low} = \begin{bmatrix} 0.175 & 0.101 \\ -0.0256 & 0.149 \end{bmatrix},$$

and for the high-beta firm we use

$$K_{high} = \begin{bmatrix} 2.38 & 0 \\ -0.755 & 1.39 \end{bmatrix} \quad Q_{high} = \begin{bmatrix} 0.175 & 0.229 \\ -0.0256 & 0.206 \end{bmatrix}.$$

Using the same market and firm-specific innovations $Z_{I,t}$, $Z_{S,t}$, and W_t, we then simulate return, variance, and covariance paths for each pair of the market index and a given equity. For each simulated path, we apply the filtering approach described in Appendix G to extract the daily betas. We then compute the average, variance, skewness, and excess kurtosis of the simulated and filtered betas for each path and report their averages across simulated paths in Table A.2. We set the risk-free rate to $r = 2.3\%$ and we set the conditional market expected return to $\mu_{I,t} = \Lambda^I \times \sigma_{I,t}^2$ where $\Lambda^I = 1.318$ which implies an unconditional expected market return equals to the 1996-2011 sample average of 6.74%. Firms' conditional expected returns are set to $\mu_{Low,t} = \Lambda^I \times \sigma_{LowI,t}$ and $\mu_{High,t} = \Lambda^I \times \sigma_{HighI,t}$ where $\sigma_{LowI,t}$ and $\sigma_{HighI,t}$ are the conditional covariances of the low and high beta firm, respectively.

Table A.2 presents the results. Comparing the first two columns of the table, we see that the mean and variance of the simulated and filtered betas are close. This indicates that our filtering approach captures well the betas of low and high beta firms.

I. The Impact of News on Beta

The covariance of beta and firm idiosyncratic risk in the model is given by

$$cov_t\left(d\beta_t; dZ_{S,t}\right) = \rho Q_I^1 \left(\frac{\sqrt{\sigma_{S,t}^2 - \beta_t^2 \sigma_{I,t}^2}}{\sigma_{I,t}}\right) dt. \tag{A.70}$$

For the empirically relevant case $\rho < 0$ (i.e. negative leverage effect) and $Q_I^1 > 0$, beta co-moves negatively with firm idiosyncratic shocks. When $dZ_{S,t}$ is large and positive, beta decreases. The covariance of beta with firm-specific innovation is more negative when the ratio of firm idiosyncratic volatility to market volatility is high.

Beta is also influenced by aggregate return risk $Z_{I,t}$. In the model, the instantaneous covariance of $d\beta_t$ and $dZ_{I,t}$ satisfies

$$cov_t\left(d\beta_t; dZ_{I,t}\right) = \frac{\rho Q_I^1}{\sigma_{I,t}} \left(\frac{Q_S^1}{Q_I^1} - \beta_t\right) dt. \tag{A.71}$$

Given that Q_S^1 and Q_I^1 should be positive in the model, the covariance of beta with aggregate shock depends on the relative level of β_t. More precisely, $cov_t\left(d\beta_t; dZ_{I,t}\right) < 0$ whenever $Q_S^1/Q_I^1 > \beta_t$ when $\rho < 0$. Empirically, we find that Q_S^1/Q_I^1 is close to 0.86. Thus, the beta of firms with relatively low betas tends to covary negatively with aggregate shocks $Z_{I,t}$. In contrast, the beta of firms with relatively large betas (i.e. $Q_S^1/Q_I^1 < \beta_t$) tends to co-move positively with $Z_{I,t}$.

Having specified a rich stochastic model for beta allows us to study how firm-specific and macro-economic news determine the cross-section of betas. To test the model implications, we rely on macro and firm-specific news indexes to proxy for aggregate return risk $dZ_{I,t}$ in (A.71) and for firm-specific shocks $dZ_{S,t}$ in (A.70). We obtain macro- and firm-level news data from January 3^{rd}, 2000 from RavenPack Analytics DowJones and Press Release files.[19] Firm-specific news reflect stories about earnings announcements, analyst forecasts, changes in credit rating, and changes in management among others. The macro-news data covers announcements about economic output, consumption, employment, foreign exchange, interest rates, credit, balance of payments, production, and housing, among others.

RavenPack converts unstructured news into an event sentiment score (ESS) which reflects whether the news is positive or negative. ESS values vary between 0 and 100. Values below

[19] We are grateful to Gunnar Grass for providing us with this data.

50, equal to 50 and above 50 correspond to negative, neutral, and positive news, respectively. The higher the score the stronger the impact of news on the firm stock price should be. To avoid repeated news, we restrict our analysis to novel news stories with a novelty score equal to 100. Furthermore, we only use news stories which are highly relevant for a given firm as indicated by a relevance score of 100.

For each firm on each day, we obtain a single value for ESS by averaging the ESS scores of all news stories observed between the previous trading day's market close and the current trading day's close. News published during non-trading days are thus assigned to the next trading day. We apply a similar methodology to macro-news in order to obtain one single ESS value per day.

Using the time-series of macro and firm-level ESS, we investigate the way beta depends on news.

I.1 The Impact of Firm-Specific News

Equation (A.70) predicts that beta and firm-specific return shocks $dZ_{S,t}$ should co-move negatively. To test this prediction, we run cross-sectional regressions. On each day, we regress the models' betas and covariances against firm-specific news controlling for lagged betas and covariances, respectively. Firm-specific news is proxied for by ESS divided by 1000. We report the average of the coefficients and their corresponding Newey-West t-statistics in Table A.3. We consider three horizons of 5, 21, and 63 days, respectively. For each horizon and variable considered, we report the time-series average of the R-squared. In all panels, we report Newey-West t-statistics computed using h' autocorrelation lags.

Table A.3 uncovers an interesting pattern. Note that on average a daily cross-section of 46 stocks have news releases. This represents about 50% of the total cross-section in our sample. Independently of the horizon considered, consistent with the model prediction, firm-specific news are negatively correlated with betas in the cross-section. Comparing the loadings on news in Panel A, B, and C, we see that there is a term structure in the coefficients. As expected, firm-level news have a much bigger impact on short term betas (resp. covariances) relative to long term betas (resp. covariances). Unlike OLS betas, our stochastic betas load significantly on news. They capture much better the information in returns which in turn justifies their significant dependence on firm-level news. OLS betas are also negatively impacted by news but do not load significantly on them. The rolling-regression OLS betas are unable to appropriately capture the impact firm-level news have on stock returns.

I.2 The Impact of Macro-News and Market Variance

Proposition 1 shows that the dynamics of beta are influenced by changes in equity covariance and market variance. Changes in market variance and firm covariance are functions of aggregate variance shocks, $W^1_{I,t}$ and $W^2_{I,t}$. As a result, we should expect that part of the co-movements in the cross-section of betas are driven by market variance risks. The model also predicts that betas covary with market return risk as seen from (A.71). To the extent that macro-news directly influence market index return risk, $Z_{I,t}$, we should expect macro-news to drive co-movements in betas. We now investigate this in detail.

For a given horizon of h' days, we run a principal component analysis on our stochastic betas and extract the first three principal components (PC) driving the cross-section of the daily betas for the 93 firms in our sample. Based on the first three PCs extracted, we construct a beta PC factor by summing up the time-series of each of the first three components times the average of the firm loadings for that component. We repeat this exercise for horizons of 5, 21, and 63 days, respectively. We perform the previous analysis for the model's forecast of integrated covariances. This allows us to analyze whether the PC factor in betas and covariances are impacted differently by aggregate return and variance risks.

Table A.4 presents the results obtained from regressing the PC factor of betas (resp. covariances) against macro-news (i.e. macro announcement ESS divided by 1000) and the VIX index. We also report the results obtained when considering the PC factor constructed from OLS betas and covariances. Each panel reports the results for a given horizon. In all panels, the t-statistics correspond to Newey-West t-statistics computed using h' autocorrelation lags. As expected, co-movements in stochastic betas and covariances are largely driven by macro-news and the VIX index. The average R-squared across horizons is 78% for stochastic betas and 72% for stochastic covariances. The PC factor of betas and covariances negatively depends on macro-news but positively loads on VIX. This result is intuitive. During market downturns, cross-sectional co-movements in betas increase. Across horizons, the coefficient loading of the PC for covariances on the VIX index is 0.20 on average. From equation (2.6), we see that $-K_{SI}$ defines the dependence of firm covariance on market variance. In Table 2, the industry average of $-K_{SI}$ is 0.27. This is close to the regression-based loading of the covariance PC on the VIX index.

Unlike the stochastic betas, the PC of OLS betas are almost completely unrelated to macro-news and the VIX index. The R-squared range from just 0.51% to 1.58%. Except at the weekly horizon, the PC of OLS betas does not load significantly on macro-news or VIX. Somewhat surprising, the R-squared obtained for the PC of OLS covariances is high. In addition, the magnitude of the coefficient estimates is comparable with the one obtained for the PC of stochastic covariances. This makes the result obtained for OLS betas striking. It is well-known that OLS betas are estimated with much noise. Because beta is the ratio of covariance to market variance, it is the ratio of two noisy estimates which is itself very noisy. This likely explains the discrepancy between the results obtained for OLS betas versus OLS covariances.

We conclude that in accordance with our model observed firm-specific and macro-news are important drivers of our stochastic betas.

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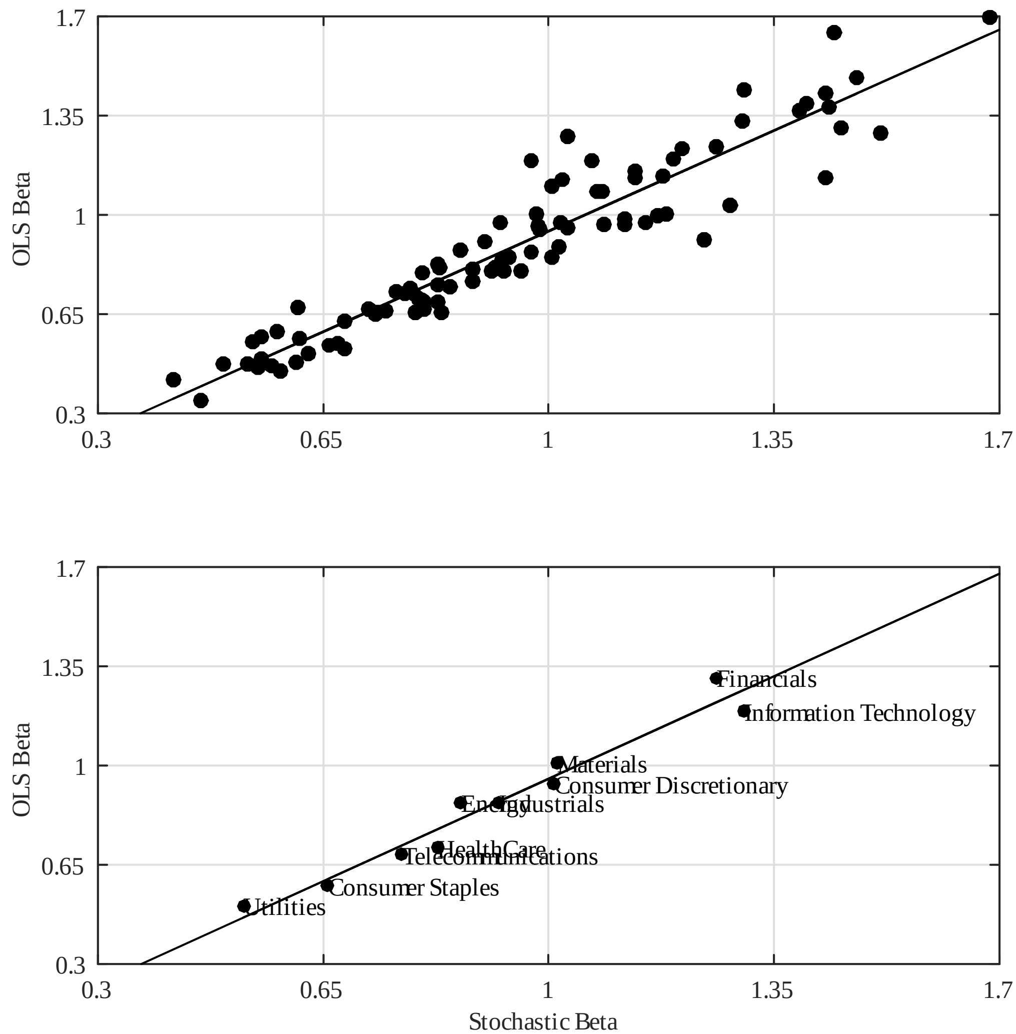
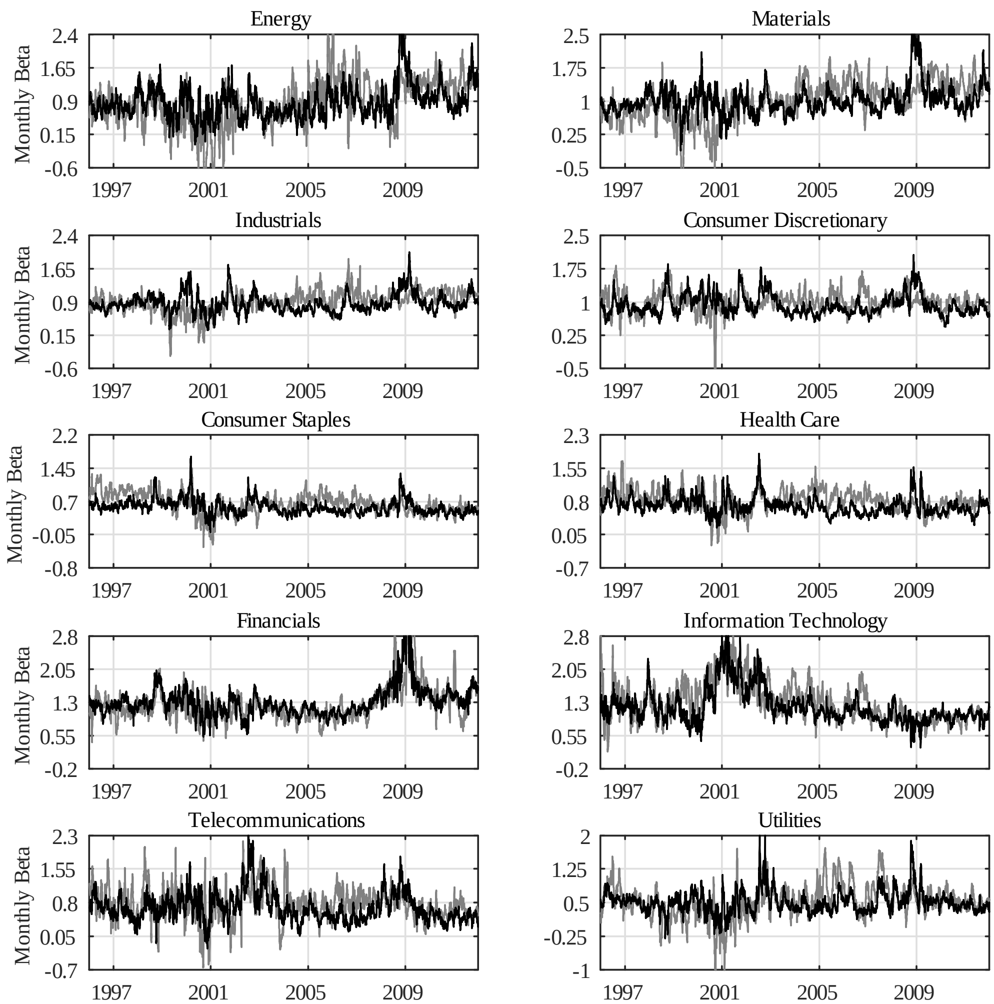
Figure 1: One-month OLS Beta versus One-month Expected Integrated Stochastic Beta, 1996-2011



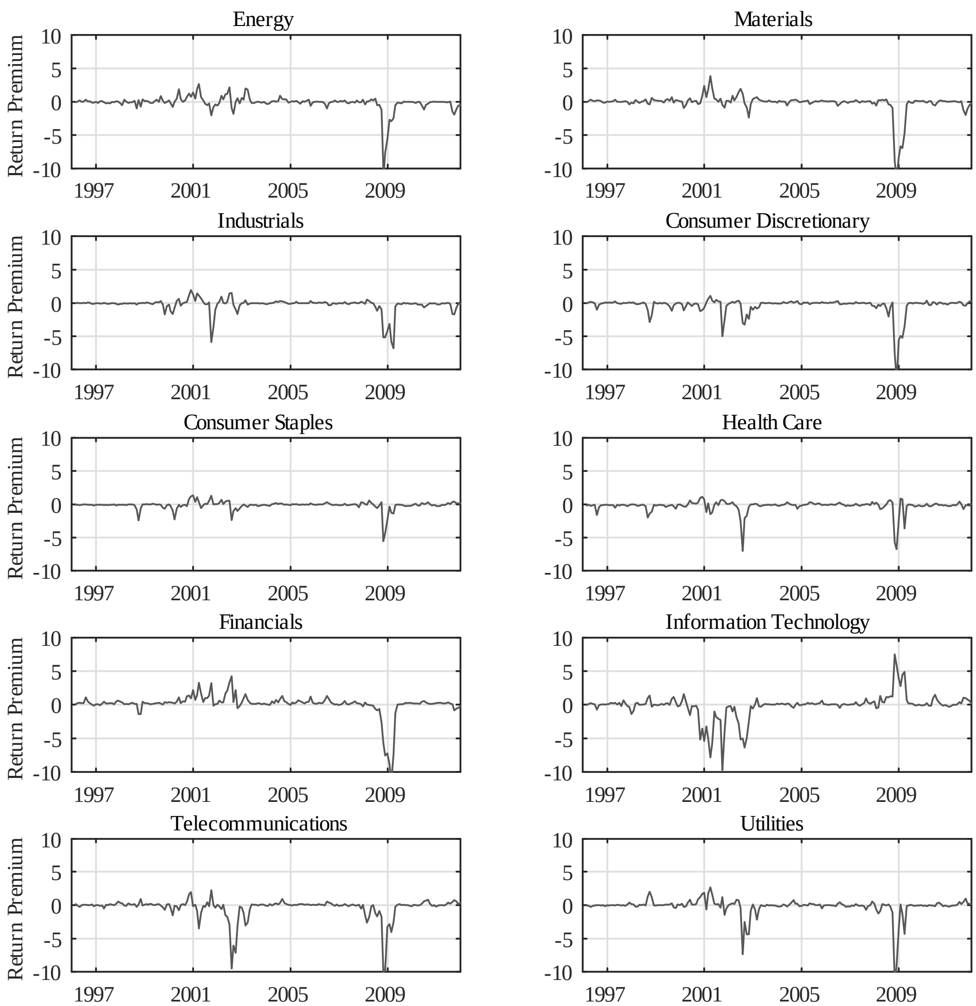
Notes to Figure: In the top panel, we scatter plot the average of the daily one-month (i.e. 21 trading days) OLS betas against the average of the daily one-month expected integrated stochastic betas for the 93 firms. In the bottom panel, we average firm level betas by industry and scatter plot industry averages of OLS betas against the stochastic expected betas.

Figure 2: One-month OLS Beta (grey)
and One-month Expected Integrated Stochastic Beta (black), 1996-2011



Notes to Figure: We plot the industry averages of the one-month OLS and stochastic expected integrated betas over time.

Figure 3: Annualized One-Month Beta Return Premia, 1996-2011



Notes to Figure: We plot the time-series of conditional one-month beta return premium by industry. For each company, we compute $RP^{BRP}_{t,\frac{21}{252}} \times \frac{252}{21} \times 100$ on each day t setting r to its sample mean of 2.3% and $\Lambda^I = 1.3176$. We then average the return premium obtained for each month and plot the average of the monthly estimates across all firms in a given industry to obtain a single time-series for each industry.

Table 1: Summary Statistics for Daily Stock Returns. 1996-2011

Index / Industry	Mean (%)	Variance (%)	OLS Beta Average	OLS Beta Minimum	OLS Beta Maximum	Systematic Risk (%)	Corre-lation	Market Cap. ($ Billions)
S&P 500	6.74	4.38						
Energy	19.61	14.51	0.94	0.80	1.15	29.26	0.54	68.44
Materials	16.34	14.40	1.04	0.90	1.14	35.80	0.59	28.08
Industrials	13.59	10.70	0.91	0.55	1.19	35.56	0.59	47.22
Cons. Discretionary	19.44	14.68	1.02	0.60	1.29	32.01	0.56	35.11
Cons. Staples	13.71	7.77	0.62	0.38	0.88	22.23	0.47	70.28
Health Care	17.04	12.73	0.77	0.57	1.05	23.75	0.48	51.60
Financials	16.33	23.36	1.47	1.06	1.96	42.92	0.65	53.11
I. T.	23.13	20.81	1.21	0.80	1.41	32.96	0.57	76.40
Telecom.	9.34	8.54	0.76	0.74	0.79	30.00	0.55	107.63
Utilities	11.98	6.25	0.53	0.40	0.60	20.16	0.45	19.01

Note to Table: The table reports summary statistics of the daily stock returns. Note that we report the average of the stock level statistics by industry. We calculate the average, variance, and various systematic risk statistics for daily stock and index returns. In the last column, we report the average market capitalization in $ billions. Average, minimum, and maximum beta, systematic risk and correlation are obtained by regressing daily excess stock returns on the S&P500 excess returns during the full sample. The sample period goes from January 4th, 1996 to December 30th, 2011.

Table 2: Parameters, Moments, and Model Fit. 1996-2011

Panel A: Model Parameters Estimates, Industry Averages

Industry	γ	K_I	K_{SI}	K_S	Q_I^1	Q_I^2	Q_S^1	Q_S^2	ρ	$\lambda_1^{\sigma_I}$	$\lambda_2^{\sigma_I}$
Energy	4.89	2.34	-0.27	1.58	0.17	-0.03	0.16	0.20	-0.81	-3.45	1.22
Materials	4.93	2.34	-0.40	1.71	0.18	-0.03	0.18	0.18	-0.81	-3.80	-0.01
Industrials	4.86	2.40	0.03	1.54	0.17	-0.02	0.15	0.17	-0.83	-3.63	1.11
Cons. Discretionary	4.96	2.39	-0.21	1.54	0.18	-0.01	0.14	0.21	-0.84	-3.80	0.48
Cons. Staples	5.00	2.43	-0.59	1.50	0.18	-0.03	0.10	0.16	-0.85	-3.53	2.12
Health Care	4.88	2.41	-0.36	1.41	0.17	-0.04	0.14	0.18	-0.85	-3.52	2.05
Financials	4.87	2.32	-0.17	1.40	0.17	0.00	0.20	0.22	-0.85	-3.97	-0.91
I. T.	4.85	2.34	-0.11	1.45	0.18	-0.02	0.20	0.21	-0.80	-3.67	0.61
Telecom.	4.98	2.40	-0.62	1.67	0.17	-0.04	0.11	0.16	-0.85	-4.11	0.80
Utilities	4.98	2.40	-0.01	1.79	0.18	-0.04	0.12	0.15	-0.85	-4.06	1.16

Panel B: Unconditional Second Moments, Risk Premiums, and Model Fit

									Log-Likelihood		
Industry	θ_I	θ_S	θ_{SI}	β_S	VRP_I	VRP_S	CRP_{SI}	BRP_S	Index Option	Equity Option	Return
Energy	0.034	0.117	0.030	0.883	-0.014	-0.010	-0.012	0.002	10,386	9,942	3,392
Materials	0.034	0.112	0.035	1.036	-0.014	-0.024	-0.019	-0.079	10,316	9,514	3,426
Industrials	0.034	0.086	0.030	0.884	-0.015	-0.013	-0.012	0.002	10,239	9,841	3,511
Cons. Discretionary	0.033	0.118	0.031	0.926	-0.014	-0.020	-0.015	-0.036	10,381	9,348	3,416
Cons. Staples	0.034	0.070	0.021	0.612	-0.014	-0.004	-0.008	0.019	10,375	10,583	3,624
Health Care	0.034	0.115	0.025	0.741	-0.014	-0.006	-0.009	0.029	10,335	9,779	3,441
Financials	0.034	0.179	0.049	1.455	-0.015	-0.066	-0.029	-0.147	10,298	7,716	3,396
I. T.	0.033	0.162	0.040	1.215	-0.013	-0.034	-0.019	-0.060	10,453	9,107	3,280
Telecom.	0.033	0.065	0.021	0.648	-0.015	-0.011	-0.012	-0.055	10,353	9,781	3,612
Utilities	0.035	0.052	0.019	0.548	-0.016	-0.005	-0.009	0.003	10,212	9,624	3,689

Note to Table: In Panel A, we report parameter estimates for the physical return dynamic. We reestimate the model for each stock paired with the S&P500 index. The parameters are obtained by maximizing the composite log-likelihood for returns and options. In Panel B, we present the unconditional moments implied by the model parameters. We also report the unconditional market and stock variance risk premiums (i.e. VRP_I and VRP_S), and the unconditional stock covariance and beta risk premiums (i.e. CRP_{SI} and BRP_S). The three last columns present log-likelihood values for returns and options.

Table 3: Leverage Effects, Beta Dynamics and Dependence Parameters. 1996-2011

Industry	ρ_I	ρ_S	Φ_{IS}	Φ_I	$\frac{\Phi_{IS}}{\Phi_I}$	λ_I^{VRP}	λ_S^{β}	$\frac{\lambda_S^{\beta}}{\lambda_I^{VRP}}$
Energy	-0.785	-0.506	0.044	0.065	0.677	-0.674	-0.318	0.472
Materials	-0.793	-0.561	0.052	0.066	0.803	-0.680	-0.669	1.011
Industrials	-0.795	-0.550	0.048	0.067	0.734	-0.723	-0.409	0.616
Cons. Discretionary	-0.821	-0.467	0.046	0.064	0.717	-0.711	-0.458	0.642
Cons. Staples	-0.829	-0.452	0.024	0.065	0.366	-0.692	0.002	-0.004
Health Care	-0.810	-0.529	0.034	0.067	0.504	-0.714	-0.118	0.141
Financials	-0.811	-0.533	0.072	0.064	1.121	-0.717	-1.008	1.436
I. T.	-0.784	-0.531	0.061	0.064	0.961	-0.669	-0.638	0.909
Telecom.	-0.823	-0.479	0.027	0.063	0.418	-0.751	-0.331	0.430
Utilities	-0.834	-0.524	0.032	0.067	0.479	-0.774	-0.298	0.388

Note to Table: The first two columns report our estimates of the market and individual equity leverage effects, ρ_I and ρ_S. We next report Φ_I and Φ_{IS} from Proposition 1, as well as their ratio. The SDF parameters λ_{VRP} and λ_β and their ratio determine the instantaneous beta risk premium. All results correspond to industry averages. The sample period goes from January 4th, 1996 to December 30th, 2011.

Table 4: Forecasting Realized Beta. 1996-2011

Model: $\beta^{OLS}_{S,t+22,21} = a_0 + a_1 \times \beta^{SB}_{S,t,21} + a_2 \times \beta^{OLS}_{S,t,21} + \varepsilon_{S,t+22,21}$

Industry	a_0	*t-Stat.*	a_1	*t-Stat.*	a_2	*t-Stat.*	Adjusted R-Squared (%)
Energy	0.389	*5.90*	0.172	*2.84*	0.410	*8.22*	22.87
Materials	0.484	*6.79*	0.111	*2.16*	0.419	*8.56*	21.47
Industrials	0.624	*9.19*	0.085	*1.32*	0.240	*4.57*	9.20
Cons. Discretionary	0.685	*8.94*	0.154	*2.11*	0.172	*3.16*	7.25
Cons. Staples	0.381	*7.24*	0.129	*1.69*	0.314	*5.71*	13.81
Health Care	0.544	*8.65*	0.130	*1.89*	0.248	*4.39*	10.32
Financials	0.488	*5.45*	0.312	*4.84*	0.293	*5.03*	23.95
I. T.	0.657	*7.87*	0.198	*3.08*	0.277	*5.21*	16.78
Telecom.	0.505	*9.52*	0.192	*3.35*	0.163	*2.74*	7.34
Utilities	0.241	*5.93*	0.230	*2.97*	0.320	*5.18*	18.53

Note to Table: The table presents the loadings and t-statistics from daily regressions of one month future realized beta on our model-implied one-month expected integrated beta controlling for lagged OLS beta. On each day, we obtain future realized beta by estimating the CAPM regression using the 21-day-ahead index and stock excess returns. We compare future realized beta with the one-month forecast from our stochastic model on that day computed using the filtered conditional beta from the previous day. The t-statistics (in italics) are calculated using Newey-West methodology with 21 lags. The sample period goes from January 4th, 1996 to December 30th, 2011.

Table 5: Predictive Cross-Sectional Regressions. Various Horizons. 1996-2011

	Dependent Variable: $R^S_{t+1,h'}$							
Horizon:	Weekly		Monthly		Quarterly		Semi-Annual	
	h' = 5 days		h' = 21 days		h' = 63 days		h' = 126 days	
	Coeff.	*t-Stat.*	Coeff.	*t-Stat.*	Coeff.	*t-Stat.*	Coeff.	*t-Stat.*
Intercept	0.0005	*0.88*	0.0035	*1.31*	0.0074	*0.91*	0.0001	*0.00*
time-*t* h'- day Expected Stochastic Beta	0.0018	*2.60*	0.0055	*1.93*	0.0179	*2.15*	0.0320	*1.70*
time-*t* h'- day OLS Beta	0.0001	*0.23*	0.0001	*0.04*	-0.0056	*-0.71*	-0.0016	*-0.08*
$R^S_{t-h'+1,h'}$	-0.0249	*-4.46*	-0.0144	*-1.08*	0.0071	*0.30*	0.0549	*1.35*
$R^S_{t-2h'+1,h'}$	-0.0136	*-2.61*	-0.0158	*-1.50*	0.0201	*0.96*	0.0225	*0.87*
$R^S_{t-3h'+1,h'}$	-0.0021	*-0.42*	0.0112	*1.12*	0.0459	*2.78*	0.0001	*0.00*
R-Squared (%)	17.58		18.10		20.78		22.00	
Newey-West Lags	5		21		63		126	

Note to Table: On each day, we regress future realized excess stock returns on stochastic and OLS betas controlling for lagged excess returns. The table reports the sample average of the daily coefficients, their t-statistics, and the average of the regression R-squares. We consider four horizons. For a given horizon of h' days, we compute future realized excess stock returns on each day by compounding the h'- day-ahead daily excess returns. The daily stochastic beta forecast (i.e. the h'- day expected integrated beta) is calculated based on the conditional beta filtered on the previous day. The daily OLS beta measure for a given horizon is obtained by regressing past excess stock returns on market excess returns using an estimation window of length equal to the horizon considered. The t-statistics in italic are calculated using the Newey-West methodology allowing for h' autocorrelation lags. The sample period goes from January 4th, 1996 to December 30th, 2011.

Table 6: Value-Weighted Portfolio Sorting Results for All NYSE stocks

Panel A: 1996-2011 Sample Period

Portfolio	$\beta_{ex-ante}$	$\beta_{ex-ante} - \beta_{ex-post}$	$\beta_{ex-post}^{HighRet}$-$\beta_{ex-post}^{LowRet}$ (High and Low based on Average Market Return)	$\beta_{ex-post}^{HighVar}$-$\beta_{ex-post}^{LowVar}$ (High and Low based on Median Squared-Market Return)	$\beta_{ex-post}^{HighVar}$-$\beta_{ex-post}^{LowVar}$ (High and Low based on Average VIX)	Ex-Post Annual Abnormal Return	$\alpha_{ex\text{-}post}$ (CAPM)	$\alpha_{ex\text{-}post}$ (FFC)
1. Low	0.40	-0.13	-4.54%	5.12%	3.18%	2.20%	2.16%	1.63%
2.	0.67	-0.07	-3.55%	3.35%	2.25%	0.24%	0.89%	-0.14%
3.	0.88	-0.04	1.26%	1.74%	3.27%	-0.26%	1.88%	-0.18%
4.	1.12	0.03	3.99%	0.13%	1.11%	-0.72%	0.69%	-0.40%
5. High	1.59	0.15	10.72%	-1.83%	-0.16%	-2.38%	-1.25%	-2.59%
H-L	1.19	0.28	15.25%	-6.95%	-3.34%	-4.58%	-3.41%	-4.22%
t-Stat.	32.90	5.39	4.54	-1.87	-1.11	-1.69	-1.40	-1.41

Panel B: 1950-2016 Sample Period

Portfolio	$\beta_{ex-ante}$	$\beta_{ex-ante} - \beta_{ex-post}$	$\beta_{ex-post}^{HighRet}$-$\beta_{ex-post}^{LowRet}$ (High and Low based on Average Market Return)	$\beta_{ex-post}^{HighVar}$-$\beta_{ex-post}^{LowVar}$ (High and Low based on Median Squared-Market Return)	$\beta_{ex-post}^{HighVar}$-$\beta_{ex-post}^{LowVar}$ (High and Low based on Average Squared-Market Return)	Ex-Post Annual Abnormal Return	$\alpha_{ex\text{-}post}$ (CAPM)	$\alpha_{ex\text{-}post}$ (FFC)
1. Low	0.36	-0.13	-5.51%	5.30%	3.46%	3.71%	2.71%	1.88%
2.	0.64	-0.08	-4.52%	3.87%	2.71%	3.35%	3.08%	1.54%
3.	0.89	-0.05	-0.31%	1.30%	1.33%	1.48%	1.88%	0.90%
4.	1.16	0.03	6.06%	-0.67%	-0.16%	-0.85%	0.05%	-0.83%
5. High	1.63	0.19	13.16%	-4.54%	-3.50%	-4.43%	-2.85%	-3.59%
H-L	1.27	0.32	18.67%	-9.85%	-6.96%	-8.14%	-5.55%	-5.47%
t-Stat.	53.95	14.82	7.92	-5.19	-5.81	-5.18	-3.79	-3.61

Note to Table: Each month, we sort stocks into quintile portfolios based on ex-ante betas obtained by regressing daily stock excess returns on daily market excess returns from the last 252 trading days. In the first column, we report the value-weighted average ex-ante betas for each portfolio. In the second column, we report the value-weighted average of the difference between ex-ante and ex-post betas, where the ex-post betas are obtained by regressing daily excess stock returns against daily excess market returns during the 252 days following the sorting. In the third column, we report the difference between a high and low market return ex-post beta (times 100), which we calculate by regressing excess stock returns against excess market returns during the next year for above- and below-average market return days, separately. In the fourth column, we report the difference between high and low squared market return ex-post betas (times 100), which we calculate by regressing excess stock returns against excess market returns during the next year for above- and below-median market squared return days separately. In the fifth column, we report the difference between high and low ex-post betas (times 100), where we use average VIX in Panel A and average market return squared in Panel B to identify high and low market variance days. In the sixth column, the ex-post annual abnormal returns are obtained for each stock by taking the difference between the compounded daily excess equity return over the next year and the product of ex-post beta and the compounded daily excess market return. The value-weighted abnormal returns are subsequently calculated for each portfolio. Finally, we report CAPM and Fama-French-Carhart (FFC) alphas for each quintile portfolio as well as for high minus low where the alphas are estimated over the full sample. The t-statistics are from Newey-West using 12 lags.

Table A.1.A: Parameter Estimates for each Stock and Index Pair, 1996-2011

Ticker	γ	K_I	K_{SI}	K_S	Q_I^1	Q_I^2	Q_S^1	Q_S^2	ρ	$\lambda_1^{\sigma_I}$	$\lambda_2^{\sigma_I}$
AAPL	5.00	2.35	-0.18	0.83	0.170	-0.045	0.173	0.207	-0.81	-3.55	1.80
ABT	5.00	2.30	0.01	1.30	0.162	-0.061	0.132	0.130	-0.87	-3.35	2.20
ADBE	4.99	2.27	-0.35	1.23	0.171	-0.028	0.227	0.214	-0.78	-3.03	1.99
ADM	4.99	2.51	-1.12	1.63	0.197	-0.030	0.095	0.199	-0.84	-4.04	2.00
ADP	4.65	2.31	-0.81	1.43	0.181	-0.012	0.087	0.157	-0.82	-3.18	4.50
AEP	5.00	2.30	-0.44	1.92	0.163	-0.059	0.136	0.134	-0.87	-3.39	2.23
AFL	4.95	2.25	-0.93	1.72	0.171	-0.022	0.211	0.180	-0.82	-3.63	0.27
AGN	5.00	2.30	-1.33	1.45	0.162	-0.061	0.124	0.178	-0.89	-3.21	2.57
AIG	5.00	2.25	-0.92	1.11	0.161	-0.089	0.304	0.188	-0.75	-3.02	1.67
ALL	4.85	2.40	1.47	1.58	0.177	0.037	0.137	0.239	-0.82	-3.21	-2.30
AMGN	4.90	2.28	-0.82	2.01	0.171	-0.036	0.158	0.228	-0.81	-2.95	1.85
APA	4.80	2.33	-0.77	1.42	0.176	-0.026	0.130	0.240	-0.84	-3.72	2.04
APC	5.00	2.27	0.29	1.40	0.158	-0.069	0.217	0.175	-0.81	-3.29	1.88
AXP	4.75	2.36	-0.12	1.48	0.179	0.027	0.165	0.221	-0.85	-3.84	0.48
BA	4.95	2.34	-0.43	1.38	0.175	-0.025	0.148	0.145	-0.77	-3.79	1.30
BAC	4.93	2.29	-0.10	0.90	0.169	0.047	0.132	0.225	-0.75	-4.38	-2.95
BAX	5.03	2.81	0.61	1.55	0.191	-0.015	0.117	0.189	-0.83	-4.18	0.58
BMY	5.04	2.45	-1.36	1.54	0.163	-0.111	0.146	0.129	-0.89	-3.09	2.80
CAT	4.90	2.32	0.53	1.83	0.176	-0.007	0.178	0.219	-0.80	-3.49	-2.16
CELG	5.00	2.29	-1.78	0.91	0.162	-0.063	0.180	0.272	-0.87	-2.84	3.39
CL	5.00	2.31	-1.01	1.51	0.165	-0.057	0.090	0.145	-0.88	-3.24	2.34
COF	4.75	2.32	-0.67	1.52	0.165	-0.069	0.321	0.172	-0.76	-4.62	-0.89
COST	4.59	2.57	0.00	0.70	0.191	-0.011	0.108	0.143	-0.85	-4.63	3.09

Note to Table: We report on parameter estimates for the Wishart P-dynamics and the price of risks. We estimate the model pairwise for each stock and the S&P500 index. The parameters are obtained by maximizing the composite log-likelihood of returns and options.

Table A.1.B: Parameter Estimates for each Stock and Index Pair, 1996-2011

Ticker	γ	K_I	K_{SI}	K_S	Q_I^1	Q_I^2	Q_S^1	Q_S^2	ρ	$\lambda_1^{\sigma_I}$	$\lambda_2^{\sigma_I}$
CSCO	4.82	2.39	0.05	0.98	0.176	-0.040	0.201	0.155	-0.83	-5.36	-2.18
CSX	5.00	2.32	-0.12	1.37	0.166	-0.054	0.177	0.141	-0.83	-3.80	0.70
CVX	5.03	2.40	-0.99	2.74	0.173	-0.040	0.140	0.185	-0.80	-3.90	0.71
DD	4.80	2.42	-1.18	1.97	0.200	-0.036	0.135	0.175	-0.82	-4.08	-0.77
DE	5.04	2.29	0.37	1.48	0.166	-0.048	0.219	0.142	-0.76	-4.04	0.59
DHR	4.75	2.37	0.00	0.89	0.181	0.017	0.097	0.153	-0.82	-3.56	4.77
DIS	4.94	2.44	0.90	2.04	0.176	0.039	0.138	0.257	-0.84	-3.80	1.77
DOW	5.00	2.35	0.05	1.89	0.169	-0.047	0.233	0.153	-0.81	-4.31	0.08
DUK	5.07	2.44	-0.20	2.16	0.196	-0.039	0.139	0.172	-0.85	-3.72	1.68
EMC	4.94	2.30	-0.41	1.56	0.173	-0.026	0.254	0.232	-0.78	-3.49	1.13
EMR	4.65	2.23	1.35	1.46	0.172	0.046	0.134	0.206	-0.87	-1.68	-5.68
EOG	5.00	2.32	-0.69	1.35	0.167	-0.053	0.157	0.208	-0.85	-3.50	2.32
EXC	4.95	2.45	0.05	1.58	0.177	-0.037	0.127	0.146	-0.86	-4.44	0.91
F	5.00	2.43	-1.31	1.50	0.177	-0.026	0.135	0.244	-0.83	-3.86	1.09
FCX	5.02	2.33	-0.71	1.15	0.173	-0.024	0.176	0.226	-0.83	-3.75	2.81
FDX	4.78	2.39	0.31	1.43	0.191	0.027	0.117	0.234	-0.82	-3.51	3.01
GE	4.95	2.53	-0.63	1.93	0.181	-0.027	0.186	0.141	-0.82	-4.53	2.59
GILD	5.01	2.28	-0.86	0.91	0.160	-0.067	0.183	0.223	-0.87	-2.77	2.38
GIS	5.92	3.39	-0.41	2.79	0.195	0.005	0.075	0.185	-0.89	-3.55	4.56
HAL	4.90	2.26	-0.43	1.64	0.170	-0.039	0.255	0.226	-0.77	-3.08	1.27
HD	4.95	2.44	1.80	1.80	0.175	0.044	0.153	0.264	-0.86	-3.06	-3.40
HON	4.95	2.45	1.24	1.59	0.176	0.038	0.146	0.241	-0.90	-3.68	-2.45
HWP	5.03	2.30	0.40	1.83	0.171	-0.029	0.227	0.223	-0.77	-3.85	1.71

Note to Table: We report on parameter estimates for the Wishart P-dynamics and the price of risks. We estimate the model pairwise for each stock and the S&P500 index. The parameters are obtained by maximizing the composite log-likelihood of returns and options.

Table A.1.C: Parameter Estimates for each Stock and Index Pair, 1996-2011

Ticker	γ	K_I	K_{SI}	K_S	Q_I^1	Q_I^2	Q_S^1	Q_S^2	ρ	$\lambda_1^{\sigma_I}$	$\lambda_2^{\sigma_I}$
IBM	4.75	2.32	0.60	1.63	0.175	0.042	0.104	0.217	-0.83	-2.41	-1.04
INTC	4.75	2.36	-0.49	1.77	0.178	-0.031	0.237	0.197	-0.74	-4.34	-1.03
JCI	4.89	2.44	-0.92	1.51	0.190	-0.045	0.159	0.129	-0.82	-4.50	-1.17
JNJ	5.00	2.35	0.27	1.49	0.170	-0.044	0.118	0.119	-0.87	-3.85	0.64
JPM	4.75	2.29	0.04	1.44	0.169	0.057	0.130	0.270	-0.93	-2.94	-2.18
KMB	4.98	2.52	0.09	0.79	0.182	-0.009	0.067	0.114	-0.86	-4.27	0.77
KO	4.80	2.29	-0.16	1.47	0.175	-0.032	0.099	0.147	-0.87	-3.61	1.27
LLY	4.34	2.63	0.81	0.76	0.199	0.009	0.120	0.160	-0.83	-4.59	1.19
LMT	5.03	2.43	-0.89	1.59	0.154	-0.117	0.142	0.124	-0.91	-2.54	3.03
LOW	4.95	2.45	0.78	1.33	0.177	0.035	0.117	0.235	-0.86	-3.43	-2.47
MCD	5.00	2.49	0.09	1.09	0.181	-0.007	0.085	0.157	-0.86	-4.58	2.07
MCK	5.00	2.34	-0.93	1.74	0.168	-0.050	0.140	0.223	-0.87	-3.28	2.10
MDT	4.71	2.44	0.22	1.10	0.183	-0.025	0.126	0.171	-0.86	-3.72	2.46
MMM	4.27	2.66	0.41	1.27	0.202	-0.009	0.133	0.160	-0.83	-4.74	4.03
MO	4.90	2.20	-1.42	2.08	0.154	-0.075	0.129	0.208	-0.87	-2.62	2.29
MRK	5.00	2.37	-1.52	1.57	0.151	-0.095	0.132	0.118	-0.91	-3.29	2.17
MS	4.96	2.39	-0.45	0.93	0.169	0.055	0.137	0.222	-0.91	-6.43	2.38
NEE	4.80	2.31	0.64	1.53	0.175	-0.030	0.122	0.131	-0.83	-3.91	0.31
NKE	5.01	2.36	-0.10	1.67	0.174	-0.030	0.143	0.213	-0.83	-3.96	2.07
NSC	4.90	2.26	-0.31	1.68	0.169	-0.041	0.210	0.131	-0.77	-3.46	0.47
ORCL	4.95	2.22	-1.15	1.67	0.173	-0.018	0.224	0.231	-0.84	-3.35	0.66
OXY	4.76	2.28	-0.45	1.19	0.176	-0.021	0.118	0.163	-0.77	-2.91	1.69
PEP	4.91	2.24	-0.57	1.06	0.160	-0.066	0.100	0.121	-0.92	-2.95	2.59

Note to Table: We report on parameter estimates for the Wishart P-dynamics and the price of risks. We estimate the model pairwise for each stock and the S&P500 index. The parameters are obtained by maximizing the composite log-likelihood of returns and options.

Table A.1.D: Parameter Estimates for each Stock and Index Pair, 1996-2011

Ticker	γ	K_I	K_{SI}	K_S	Q_I^1	Q_I^2	Q_S^1	Q_S^2	ρ	$\lambda_1^{\sigma_I}$	$\lambda_2^{\sigma_I}$
PFE	4.65	2.47	0.45	1.21	0.187	-0.005	0.130	0.180	-0.79	-4.16	-0.05
PG	4.95	2.19	-0.89	1.99	0.169	-0.029	0.129	0.149	-0.75	-3.09	1.55
PNC	4.95	2.44	1.76	1.52	0.174	0.052	0.165	0.286	-1.00	-3.55	-3.75
PX	4.91	2.27	-0.70	1.71	0.168	-0.050	0.165	0.133	-0.79	-3.35	0.00
QCOM	4.44	2.55	1.56	0.95	0.194	-0.003	0.214	0.269	-0.80	-3.92	-2.10
RTN	4.80	2.32	-1.36	1.92	0.152	-0.093	0.161	0.197	-0.88	-3.03	2.56
SBUX	5.00	2.32	-0.78	1.12	0.165	-0.055	0.162	0.186	-0.87	-3.56	1.98
SLB	4.85	2.45	-0.73	1.02	0.179	-0.033	0.126	0.168	-0.82	-3.81	2.50
SO	5.08	2.52	-0.08	1.79	0.180	-0.011	0.070	0.167	-0.87	-4.81	0.65
STT	4.85	2.25	-1.82	1.84	0.171	-0.047	0.255	0.185	-0.90	-4.11	-1.82
SYK	4.40	2.62	0.70	1.14	0.198	0.000	0.123	0.202	-0.84	-4.20	2.44
T	4.95	2.46	-0.37	1.69	0.179	-0.026	0.112	0.172	-0.84	-4.58	0.38
TGT	4.90	2.27	-0.35	2.00	0.170	-0.038	0.192	0.217	-0.81	-3.58	0.53
TJX	5.02	2.22	-0.97	1.24	0.163	-0.051	0.138	0.180	-0.85	-2.87	2.15
TMO	5.05	2.34	-0.27	2.37	0.183	-0.064	0.213	0.194	-0.83	-3.71	1.07
TWX	4.95	2.49	-1.43	1.61	0.181	-0.010	0.151	0.278	-0.81	-4.57	0.60
TXN	5.00	2.33	-0.46	2.03	0.168	-0.051	0.265	0.249	-0.78	-3.88	1.30
UNH	5.03	2.30	-0.36	1.98	0.174	-0.066	0.209	0.240	-0.85	-3.16	2.29
UNP	4.95	2.48	0.23	1.44	0.182	-0.006	0.121	0.173	-0.84	-4.26	2.64
UTX	4.88	2.47	0.27	1.46	0.182	0.015	0.116	0.179	-0.83	-4.17	1.69
VZ	5.00	2.34	-0.88	1.65	0.168	-0.049	0.110	0.151	-0.85	-3.63	1.22
WAG	5.01	2.29	-0.53	1.31	0.170	-0.034	0.103	0.157	-0.84	-3.38	2.43
WMT	4.99	2.25	-0.44	1.18	0.172	-0.008	0.081	0.153	-0.82	-3.43	0.39
XON	4.83	2.39	1.59	1.87	0.198	0.025	0.160	0.197	-0.81	-3.39	-2.62

Note to Table: We report on parameter estimates for the Wishart P-dynamics and the price of risks. We estimate the model pairwise for each stock and the S&P500 index. The parameters are obtained by maximizing the composite log-likelihood of returns and options.

Table A.2: Simulation Results

	Stochastic Model Betas			
	Simulated		Particle Filtered	
	Low Beta	High Beta	Low Beta	High Beta
Average	0.602	1.610	0.577	1.564
Variance	42.85%	121.73%	27.02%	83.64%
Skewness	-0.021	0.001	0.202	0.037
Kurtosis	8.347	8.588	2.546	2.288

Note to Table: We simulate 10,000 paths of daily returns and betas of two hypothetical firms and the market index over ten years based on the model dynamics. We set the market model parameters to the cross-sectional average of the market parameters estimated in our empirical analysis. The remaining parameters are then defined to generate an unconditional beta of 0.60 (i.e. low beta firm) or 1.60 (i.e. high beta firm). Based on these values, we simulate return paths of the index and a given equity. For each simulated path, we apply the filtering approach described in Appendix G and filter the daily betas taking the index and equity structural parameters as given. We compute the average, annualized variance, skewness, and excess kurtosis of the simulated and filtered betas for each path and report the average across simulations of the four moments obtained in the table. For simulation purpose, we set the risk-free rate to r =2.3% and the conditional market expected return to $\mu\,(_{I,t}) = 1.3176\times\sigma^2(_{I,t})$ which implies an unconditional expected market return equal to the 1996-2011 sample average of 6.74%. Moreover, the conditional expected return of the two firms are $\mu\,(_{Low,t}) = 1.3176\times\sigma(_{LowI,t})$ and $\mu\,(_{High,t}) = 1.3176\times\sigma(_{HighI,t})$ where $\sigma(_{LowI,t})$ and $\sigma(_{HighI,t})$ are the conditional covariances of the low and high beta firm, respectively.

Table A.3: Firm-Specific News and the Cross-section of Betas and Covariances. 2000-2011

Panel A: The Cross-Section of Stochastic and OLS Betas and Covariances, Weekly Horizon

Dependent Variable: Cross-Section of	1 week Exp. Int. Stochastic $\beta^{SB}_{S,t,5}$		1 week Exp. Int. Stochastic $\sigma^{SB}_{SI,t,5}$		1 week OLS $\beta^{OLS}_{S,t,5}$		1 week OLS $\sigma^{OLS}_{SI,t,5}$	
	Coeff.	*t-Stat.*	Coeff.	*t-Stat.*	Coeff.	*t-Stat.*	Coeff.	*t-Stat.*
Intercept	0.1488	*18.28*	0.0052	*14.04*	0.7329	*28.68*	0.0348	*11.28*
$FirmNews_{j,t}$	-1.0712	*-10.43*	-0.0366	*-9.41*	-0.4189	*-1.07*	-0.0405	*-2.17*
$\beta^{SB/OLS}_{S,t-4,5}$ / $\sigma^{SB/OLS}_{SI,t,t-4,5}$	0.9071	*207.64*	0.9091	*154.05*	0.2526	*24.18*	0.2941	*18.72*
R-Squared (%)	81.19		80.75		14.75		14.75	
Newey-West Lags	5		5		5		5	

Panel B: The Cross-Section of Stochastic OLS Betas and Covariances, Monthly Horizon

Dependent Variable: Cross-Section of	1 mth Exp. Int. Stochastic $\beta^{SB}_{S,t,21}$		1 mth Exp. Int. Stochastic $\sigma^{SB}_{SI,t,21}$		1 mth OLS $\beta^{OLS}_{S,t,21}$		1 mth OLS $\sigma^{OLS}_{SI,t,21}$	
	Coeff.	*t-Stat.*	Coeff.	*t-Stat.*	Coeff.	*t-Stat.*	Coeff.	*t-Stat.*
Intercept	0.2888	*14.53*	0.0105	*11.47*	0.4267	*18.50*	0.0189	*4.81*
$FirmNews_{j,t}$	-1.0287	*-5.78*	-0.0360	*-5.51*	-0.0415	*-0.22*	-0.0164	*-1.09*
$\beta^{SB/OLS}_{S,t-20,21}$ / $\sigma^{SB/OLS}_{SI,t,t-20,21}$	0.7528	*60.06*	0.7671	*42.09*	0.5551	*27.15*	0.6583	*14.44*
R-Squared (%)	57.54		57.64		35.93		35.93	
Newey-West Lags	21		21		21		21	

Panel C: The Cross-Section of Stochastic and OLS Betas and Covariances, Quarterly Horizon

Dependent Variable: Cross-Section of	3 mth Exp. Int. Stochastic $\beta^{SB}_{S,t,63}$		3 mth Exp. Int. Stochastic $\sigma^{SB}_{SI,t,63}$		3 mth OLS $\beta^{OLS}_{S,t,63}$		3 mth OLS $\sigma^{OLS}_{SI,t,63}$	
	Coeff.	*t-Stat.*	Coeff.	*t-Stat.*	Coeff.	*t-Stat.*	Coeff.	*t-Stat.*
Intercept	0.3320	*11.11*	0.0119	*11.12*	0.2902	*8.12*	0.0147	*2.42*
$FirmNews_{j,t}$	-0.7984	*-3.60*	-0.0284	*-3.62*	-0.1005	*-0.71*	-0.0192	*-1.34*
$\beta^{SB/OLS}_{S,t-62,63}$ / $\sigma^{SB/OLS}_{SI,t,t-62,63}$	0.6994	*31.56*	0.7459	*20.44*	0.7121	*20.18*	0.8616	*9.17*
R-Squared (%)	50.02		50.62		51.51		51.51	
Newey-West Lags	63		63		63		63	

Note to Table: The table presents the cross-sectional result of regressing betas and covariances against the firm-specific news index (ESS/1000) and lagged variables. The daily stochastic beta/covariance forecast (i.e. h'-day expected integrated beta/covariance) are calculated based on the spot variables filtered on the previous day. The daily OLS beta/covariance measure for a given horizon is computed from past excess returns using an estimation window of length equal to the horizon considered. The coefficients reported are Fama-MacBeth coefficients. We consider three horizons. The t-statistics in italic are calculated using the Newey-West methodology allowing for h' autocorrelated lags in the residuals. The sample period is from January 3rd, 2000 to December 30th, 2011. The regression frequency is daily.

Table A.4: Macro-News, Market Variance, and Co-movements in Betas and Covariances. 2000-2011

Panel A: Principal Component of Stochastic and OLS Betas and Covariances, Weekly Horizon

Dependent Variable: PC Factor of	1 week Exp. Int. Stochastic $\beta^{SB}_{S,t,5}$		1 week Exp. Int. Stochastic $\sigma^{SB}_{SI,t,5}$		1 week OLS $\beta^{OLS}_{S,t,5}$		1 week OLS $\sigma^{OLS}_{SI,t,5}$	
	Coeff.	*t-Stat.*	Coeff.	*t-Stat.*	Coeff.	*t-Stat.*	Coeff.	*t-Stat.*
Intercept	0.5745	*52.59*	-0.0087	*-4.62*	0.9575	*66.27*	-0.0927	*-4.92*
$MacroNews_t$	-0.3662	*-2.42*	-0.1373	*-5.60*	0.0720	*0.35*	-0.4788	*-3.68*
Vix_t	1.3901	*37.67*	0.2364	*32.87*	-0.0854	*-2.27*	0.7422	*7.52*
R-Square (%)	75.30		71.77		0.51		44.06	
Newey-West Lags	5		5		5		5	

Panel B: Principal Component of Stochastic and OLS Betas and Covariances, Monthly Horizon

Dependent Variable: PC Factor of	1 mth Exp. Int. Stochastic $\beta^{SB}_{S,t,21}$		1 mth Exp. Int. Stochastic $\sigma^{SB}_{SI,t,21}$		1 mth OLS $\beta^{OLS}_{S,t,21}$		1 mth OLS $\sigma^{OLS}_{SI,t,21}$	
	Coeff.	*t-Stat.*	Coeff.	*t-Stat.*	Coeff.	*t-Stat.*	Coeff.	*t-Stat.*
Intercept	0.5903	*38.81*	-0.0045	*-1.92*	0.9576	*55.91*	-0.0959	*-4.69*
$MacroNews_t$	-0.4464	*-2.72*	-0.1218	*-4.84*	0.1138	*0.64*	-0.2307	*-2.11*
Vix_t	1.3915	*22.23*	0.2113	*20.51*	-0.0816	*-1.31*	0.6928	*5.81*
R-Square (%)	77.17		71.78		0.95		65.07	
Newey-West Lags	21		21		21		21	

Panel C: Principal Component of Stochastic and OLS Betas and Covariances, Quarterly Horizon

Dependent Variable: PC Factor of	3 mth Exp. Int. Stochastic $\beta^{SB}_{S,t,63}$		3 mth Exp. Int. Stochastic $\sigma^{SB}_{SI,t,63}$		3 mth OLS $\beta^{OLS}_{S,t,63}$		3 mth OLS $\sigma^{OLS}_{SI,t,63}$	
	Coeff.	*t-Stat.*	Coeff.	*t-Stat.*	Coeff.	*t-Stat.*	Coeff.	*t-Stat.*
Intercept	0.6318	*33.85*	0.0040	*1.68*	0.9633	*37.53*	-0.0745	*-3.89*
$MacroNews_t$	-0.4718	*-2.79*	-0.0946	*-4.04*	0.0949	*0.39*	-0.1597	*-1.68*
Vix_t	1.2965	*15.24*	0.1607	*15.54*	-0.0998	*-1.05*	0.5778	*5.01*
R-Square (%)	79.27		72.03		1.58		63.46	
Newey-West Lags	63		63		63		63	

Note to Table: We regress the first principal component of the models' betas and covariances against the macro-economic news index and VIX for three horizons. The daily stochastic beta/covariance forecast (i.e. h'-day expected integrated beta/covariance) are calculated based on the spot variables filtered on the previous day. The daily OLS beta/covariance measures for a given horizon are computed from past excess returns using an estimation window of length equal to the horizon considered. For each variable, the principal component factor is obtained by summing the first three principal components multiplied by the average loading of stocks for that component. The t-statistics in italics are calculated using the Newey-West methodology allowing for h' autocorrelated lags in the residuals. The sample period is from January 3rd, 2000 to December 30th, 2011. The regression frequency is daily.