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An efficient method to price counterparty risk

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This working paper has been written by

Michèle Breton
GERAD and HEC Montréal

Oussama Marzouk
HEC Montréal

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Michèle Breton
GERAD and HEC Montréal

Oussama Marzouk
HEC Montréal

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Abstract

We introduce an efficient approach to evaluate counterparty risk and compute the Credit Value Adjustement for derivatives having early exercise features. The approach is flexible and can account for Wrong Way Risk and various models for the underlying asset's dynamics. Numerical experiments are presented to illustrate the efficiency of the method.

1 Introduction

Counterparty risk is defined as the risk faced by an entity entering an Over-the-counter (OTC) contract with a counterparty having a relevant default probability, so that it might not respect its payment obligations. Since the global financial crisis of 2008, it has been recognized that credit risk should be taken into account when evaluating the value of OTC derivatives. The *Credit Value Adjustment* (CVA) capital charge required by Basel III incorporates the impact of counterparty risk, as it measures the expected potential loss from counterparty default. *Wrong Way Risk* (WWR) is the additional risk faced when the underlying asset and the default of the counterparty are correlated.

While the computation of CVA is straightforward for European-style derivatives (Klein 1996, Gregory 2010), it is not the case for derivatives with early exercise opportunity because the CVA is then path-dependent (exposure falls to zero after exercise). In that case, the CVA is presently evaluated using computationnally intensive simulation-based methods.

In this paper, we introduce a new approach to price counterparty risk and compute the CVA for derivatives having early exercise features. This approach is much more efficient than presently available methods, providing an accurate evaluation without the need for costly simulation. Moreover, the algorithm provides more than a point estimate: it yields the value of the vulnerable derivative and the CVA for all possible values of the underlying asset and time to maturity. The CVA pricing model is implemented using an intensity model for counterparty default, which can be calibrated to market data.

The paper is organized as follows. In Section 2, we briefly present the counterparty risk pricing framework and review some existing approaches for European options, American options, and WWR. Section 3 presents a general CVA pricing model using a discrete dynamic programming framework in the context of a Bermudian put option. Section 4 reports on numerical experiments with different market and correlation models. Section 5 is a conclusion.

2 Counterparty risk pricing

This paper examines counterparty risk from a pricing point of view, that is, how should the price of instruments be adjusted to account for possible default of the counterparty. The so-called CVA is defined as the value of this adjustment, and corresponds to the expected potential loss due to a future default of the counterparty.

More specifically, consider a defaultable claim maturing at T. Denote by N_t the net cash flows of the claim between t and T, discounted back at t, from the point of view of the investor facing counterparty risk. Denote by V_t^D the value of this defaultable claim at t, and by V_t the value of an equivalent counterparty-risk-free claim at t. At valuation time t, provided the counterparty has not defaulted before t, the general counterparty risk pricing formula is given by

$$V_t^D = V_t - (1 - R)\, \mathbb{E}_t[1_{\{t<\tau\leq T\}} \beta_t\,(\tau)\,[N_\tau]^+] \tag{1}$$

where τ is the default time, $\beta_t\,(\tau)$ is the risk-free discount factor between dates t and τ, and R is the recovery factor, assumed deterministic. The notation $\mathbb{E}_t\,[\cdot]$ represents the expectation, under the risk-neutral measure, conditional on the information up to t, including default monitoring. The functions 1_A and $[x]^+$ are defined respectively by

$$\begin{aligned} 1_A &= \begin{cases} 1 \text{ if } A \\ 0 \text{ otherwise} \end{cases} \\ [x]^+ &= \max\{0, x\}. \end{aligned}$$

The corrective term in formula (1) is the Credit Value Adjustment at date t:

$$\begin{aligned} CVA_t &= (1-R)\,\mathbb{E}_t[1_{\{t<\tau\leq T\}}\beta_t(\tau)\,[N_\tau]^+] \\ &= V_t - V_t^D. \end{aligned} \tag{2}$$

In the following paragraphs, we briefly describe our model for counterparty default time and present a simple formula that can be used for European derivatives under the assumption of independent random events. We then review existing approaches for derivatives with early exercise opportunities and for the case where the default event is correlated with the underlying portfolio value. For more details about counterparty risk features, see Brigo (2011) and Gregory (2010).

2.1 Counterparty default model

A number of approaches have been developed to incorporate the impact of credit risk on the value of derivatives. These approaches can be divided into two major categories, according to the way the default event is modeled, that is, either using structural or intensity models.

In *structural models,* the event of default of a given firm is related to the evolution of some of its structural variables. This approach originated with the paper of Merton (1974) where the firm defaults if the value of its assets is below its outstanding debt at maturity. Several extensions of this model have been developed; for instance, the Black and Cox (1976) model, in which the firm's default occurs as soon as the firm's asset value drops to a given default threshold. Examples of papers that use structural models to price contingent claims subject to default risk include Johnson and Stulz (1987) and Klein (1996).

On the other hand, in *intensity models*, default is treated as an unexpected event whose likelihood is governed by a default-intensity process. Examples of intensity models include Jarrow and Turnbull (1995), Duffie and Huang (1996), and Jarrow and Yu (2001).

One fundamental difference between intensity and structural models is that intensity models use only public (market) information, which is fully observable. As such, the relation between default and the firm value is not considered explicitly. It is therefore argued that intensity models are more useful for investors who use them for pricing and hedging, whereas structural models are more appropriate for managers and regulators. For a more detailed discussion about the two categories of model, see Benito et al. (2005).

In this paper, we use a simple intensity model where default is assumed to obey a homogeneous Poisson process. The assumptions underlying this model is that the probability of default in a very small time interval, conditional on no prior default before that time, is constant over time. The constant hazard rate, denoted λ, is called the *intensity of default.* The default time τ is the first jump time of this Poisson process. In such a model, two parameters are sufficient to characterize counterparty risk, namely the recovery factor R and the intensity of default λ. These parameters can be estimated using market data, for instance ratings from Credit Rating Agencies and spreads of Credit Default Swaps (CDS). Under the homogeneous Poisson assumption, it can be shown (see Brigo and Masetti 2005 and Gregory 2010) that

$$\lambda (1 - R) \approx \kappa, \tag{3}$$

where κ is the (continuous) rate which makes the premium leg of a CDS equivalent to its protection payment, or loss given default $1 - R$. Usually, the parameter R is estimated using historical category data published by rating agencies; equation (3) can then be used to estimate the parameter λ using CDS spread data.

2.2 A simple formula for European options

In the case of European options, there is a single positive cashflow paid at maturity. If the default time and the evolution of the underlying asset are independent, it is then easy to show that expression (2) reduces to

$$CVA_t = (1 - R) V_t F_t (T)$$

where $F_t (T)$ is the probability of default before maturity T, conditional to no prior default before t, and V_t is the standard risk-neutral price of a European option, offered by a default-free counterparty, with the same characteristics as the vulnerable one. Assuming a homogeneous Poisson process of intensity λ, for any time $u \geq t$,

$$F_t(u) = 1 - e^{-\lambda(u-t)}$$

so that the CVA and the value of the defaultable option are given respectively by

$$\begin{aligned} CVA_t &= (1-R)\, V_t \left(1 - e^{-\lambda(T-t)}\right) \\ V_t^D &= \left(R + (1-R)\, e^{-\lambda(T-t)}\right) V_t. \end{aligned}$$

The value of a defaultable security is expressed as a fraction of the default-free value, which depends on the default parameters λ and R and on the time to maturity. In particular, if $R = 0$, we get $V_t^D = e^{-\lambda(T-t)} V_t$ where is apparent that the intensity of default appears as a correction in the discount rate of the final payoff.

2.3 Early exercise opportunities

For derivatives with early exercise opportunities, the approach presented above is no longer applicable, because the dates of the cash-flows are not known; the exercise date, after which the exposure falls to zero, depends on the path of the underlying asset and on the exercise strategy. On the other hand, the loss upon default depends on the expected value of future cash flows. The evaluation of the CVA is presently performed using simulation approches involving two random stopping times, that is, the exercise date and the default date.

Such approaches consist in first evaluating the value function and exercise strategy of the default-free option using an efficient numerical method (binomial tree, finite differences, Least Square Monte-Carlo, or dynamic programming), and then jointly simulating default and asset price trajectories to compute the eventual loss on each simulation path. Losses, corresponding to a deterministic fraction of the option value upon default if the option has not yet been exercised, are then averaged to obtain the CVA. Simulation approaches are computationally intensive (see Kan 2010 and Lu & Juan 2011).

2.4 Wrong Way Risk

Wrong Way Risk is the additional risk faced by the investor when the underlying portfolio and the default of the counterparty are correlated in the worst possible way. For instance, consider a European put option written on some underlying asset, and suppose that the underlying asset's value is positively correlated with the counterparty default time. If the price of the asset goes down dramatically, then both the exposure (because of the increase in the payoff at maturity) and the probability of default (because of the decrease in expected default time) increase. Intuitively, the presence of WWR should increase the CVA. The opposite situation (Right Way Risk) indicates a beneficial relationship between exposure and default probability that actually reduces counterparty risk. This would be the case for a European call option on the same underlying asset, which would be in the money when credit quality is higher. Figure 1 below, reproduced from Gregory (2010), illustrates the impact of correlation between the default time and the level of the underlying asset price on the value of European puts and calls, where it is apparent that the impact of Right Way Risk is far less dramatic than that of WWR.

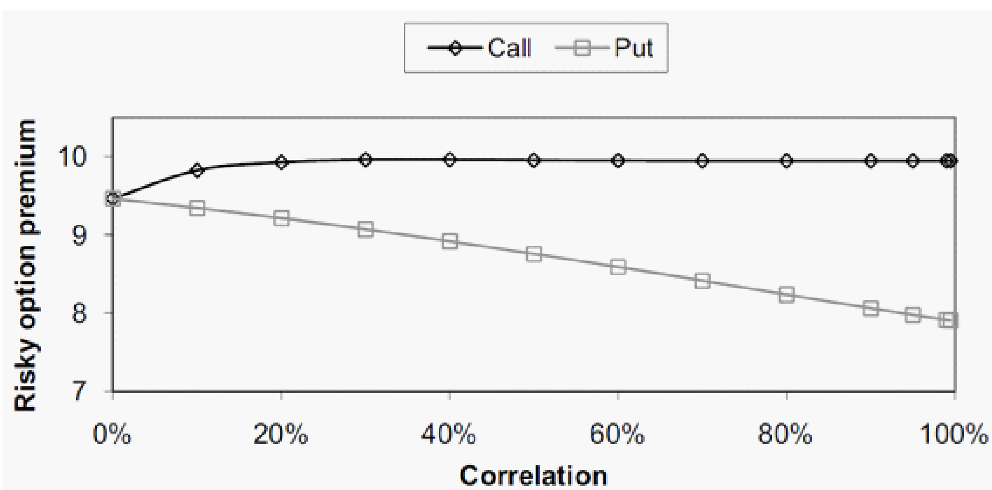

Figure 1: Value of vulnerable ($R = 0$ and $\lambda = 0.05$) European call and put as a function of correlation. Asset price is 100, exercise price is 105.1, riskless interest rate is 5% and maturity is 1 year. (Source: Gregory (2010))

In the literature, there are mainly two approaches to incorporate this correlation into the pricing model. The first one involves linking the time evolution of the underlying asset with the default time τ. Gregory (2010) adopts this framework and uses Gaussian copulas to model the correlation between the underlying asset and the default time. It is then possible to develop analytical expressions for the expected exposure, the CVA and the value of a European risky option, similarly as in Section 2.2; however, these formulas are not in closed-form and involve the numerical integration of a complex fonction.

The second approach involves linking an auxiliary variable, having a consistent relationship with the derivative portfolio, with the intensity of default, so that the hazard rate becomes stochastic over time. This type of approach is adopted in Hull & White (2012). The two types of correlation models can be easily incorporated both in simulation-based approaches and in the approach we propose in this paper.

3 A general CVA pricing model

We now present a general model for the numerical evaluation of counterparty risk, which can be used whenever closed-form formulas are not available. To simplify the exposition, we first consider the case of a vulnerable Bermudian put option written on a single asset, where the time to default and the asset price are not correlated.

3.1 The risk-free case

Consider a Bermudian put option of maturity T written on the underlying asset, not subject to counterparty risk. This option gives its holder the right to sell the asset for a predetermined *strike price* K at any date in a set of M equally spaced *exercise dates* $t_m = m\Delta$, $m = 1, ..., M$, where $\Delta \equiv \frac{T}{M}$ is the elapsed time between two successive exercise dates, and where $t_0 = 0$ is the inception date of the contract. Notice that the European option corresponds to $M = 1$, while the American option corresponds to the limit when $M \to \infty$. Let the price of the underlying asset be a Markov process $\{S_t\}$ that verifies the fundamental no-arbitrage property, where $S_t \in [0, +\infty)$. The value of the option at any date t_m, when the price of the underlying asset $S_{t_m} = s$, is given by:

$$v_m(s) = \begin{cases} h_m(s) & \text{for } m = 0 \\ \max\left\{[K-s]^+, h_m(s)\right\} & \text{for } m = 1, ..., M-1 \\ [K-s]^+ & \text{for } m = M \end{cases} \tag{4}$$

where $[K-s]^+$ is the *exercise payoff* of the option at s and where $h_m(s)$ denotes the *holding value* of the option at t_m and s. Under standard no-arbitrage assumptions, the discounted price of the underlying asset is a martingale with respect to the risk-neutral measure, and the holding value, representing the expected value of the future potentialities of the contract, is given by:

$$h_m(s) = \beta \mathbb{E}_{m,s}[v_{m+1}(S_{t_{m+1}})] \quad \text{for } m = 0, ..., M-1, \tag{5}$$

where $\beta \equiv e^{-r\Delta}$ is the discount factor between two exercise dates associated with the risk-free rate r, assumed deterministic and constant, and $\mathbb{E}_{m,s}[.]$ denotes the expectation with respect to the risk-neutral measure, conditional to $S_{t_m} = s$. Equations (4)-(5) characterize the value of the option if it is exercised optimally: at each exercise date, the option is exercised if the exercise payoff is higher than the value of the future potentialities of the contract.

Assume that v_{m+1} is known over $[0, +\infty)$, and that the market model specifies the conditional distribution of $S_{t_{m+1}}|S_{t_m}$, so that the expectation in (5) can be obtained, either numerically or analytically, for any s. Consequently, the value of $h_m(s)$ and $v_m(s)$ can be obtained for any s, and more specifically can be evaluated on a finite set of grid points. Interpolation techniques can then be used to approximate v_m by a continuous function $\hat{v}_m$, defined over $[0, +\infty)$.

The dynamic programming (DP) approach consists in evaluating the value function v_m for all m by backward induction. Thus, assume that, at stage $m+1 \leq M$, the option value as a function of the underlying asset price is described by a function $\hat{v}_{m+1}$ defined on $[0, +\infty)$. At stage m, the approximate DP algorithm consists in evaluating on a set of $p+1$ grid points a_i, $i = 0, ..., p$,

$$\tilde{h}_m(i) = \beta \mathbb{E}_{m,a_i}[\hat{v}_{m+1}(S_{t_{m+1}})] \tag{6}$$

$$\tilde{v}_m(i) = \begin{cases} \tilde{h}_m(i) & \text{for } m = 0 \\ \max\left\{[K-a_i]^+, \tilde{h}_m(i)\right\} & \text{for } m = 1, ..., M-1 \\ [K-a_i]^+ & \text{for } m = M. \end{cases} \tag{7}$$

An interpolation function $\hat{v}_m$ coinciding with the values $\tilde{v}_m(i), i = 0, ..., p$ and defined on $[0, +\infty)$ is then obtained. Starting from the known function $\hat{v}_M$, the process is repeated until $\hat{v}_0$ is determined.

For many market models, the use of polynomial or piecewise polynomial functions allows for a closed-form expression of the expectation in (6). In our numerical experiments, we interpolate the integrand in (6) using Chebyshev polynomials with Gauss-Lobato grid points, which we found to be the most efficient implementation. With this kind of interpolation, the DP algorithm can be applied with any market model, provided that the conditional density of $S_{t+1}|s$ can be computed efficiently. For more details on the use of approximate DP and polynomial interpolation for option valuation, see Breton & de Frutos (2012).

We now show how to adapt the DP algorithm to evaluate the CVA of a vulnerable option under two alternative assumptions about the behavior of the option holder.

3.2 Farsighted option holder

Consider a vulnerable Bermudian option with the same characteristics as the risk-free option described in Section 3.1. In our first formulation, we assume that the option holder, when comparing the exercise payoff to the holding value, is aware that he may not recover his claim in the future. For this farsighted investor, the holding value, representing the future potentialities of the option, is that of a vulnerable option, and is therefore adjusted for the possibility of default. Denote by $v_m^D(s)$ the value of this option at t_m when the price of the underlying asset is s. At a given exercise date t_m, provided that the counterparty has not defaulted yet, the option holder can either exercise immediately or hold the option until at least the next exercise date. The holding value, adjusted for the possibility of default between dates t_m and t_{m+1} is then

$$\begin{aligned} h_m^D(s) &= \beta \mathbb{E}_{m,s}[1_{\{\tau > t_{m+1}\}} v_{m+1}^D \left(S_{t_{m+1}}\right)] + \beta \mathbb{E}_{m,s}[1_{\{t_m < \tau \le t_{m+1}\}} R v_{m+1}^D \left(S_{t_{m+1}}\right)] \\ &= \beta \mathbb{E}_{m,s}[v_{m+1}^D \left(S_{t_{m+1}}\right)] - A_m(s) \end{aligned}$$

where, as in (2), the adjustement is given by

$$A_m(s) = (1 - R) \beta \mathbb{E}_{m,s}[1_{\{t_m < \tau \le t_{m+1}\}} v_{m+1}^D \left(S_{t_{m+1}}\right)].$$

Under our intensity model, and assuming independence between counterparty default and asset price, we readily obtain as in Section 2.2:

$$
\begin{aligned}
A_m(s) &= (1-R)\left(1-e^{-\lambda\Delta}\right)\beta\mathbb{E}_{m,s}[v^D_{m+1}\left(S_{t_{m+1}}\right)] \\
h^D_m(s) &= \beta'\mathbb{E}_{m,s}[v^D_{m+1}\left(S_{t_{m+1}}\right)].
\end{aligned}
$$

where $\beta' = \beta\left(R+(1-R)\,e^{-\lambda\Delta}\right)$. The DP algorithm to compute the value of the vulnerable option is then given by

$$
\begin{aligned}
v^D_m(s) &= \begin{cases} h^D_m(s) & \text{for } m=0 \\ \max\left\{[K-s]^+, h^D_m(s)\right\} & \text{for } m=1,...,M-1 \\ [K-s]^+ & \text{for } m=M \end{cases} \\
h^D_m(s) &= \beta'\mathbb{E}_{m,s}[v^D_{m+1}(S_{t_{m+1}})], \quad \text{for } m=0,...,M-1\,.
\end{aligned}
$$

The computation of the value of a vulnerable option is done exactly in the same way as in Section 3.1, except for the adjustement in the discount factor. The CVA can be obtained at any exercise date as a function of the asset price by computing the difference

$$CVA_m(s) = v_m(s) - v^D_m(s).$$

Because the decision to hold or to exercise the option is taken by comparing the exercise payoff to the vulnerable value v^D_m, the exercise strategy in this first model is different from the exercise strategy of the corresponding counterparty-risk-free option.

3.3 Myopic option holder

In existing models for the evaluation of the CVA, it is however usually assumed that the exercise strategy for a vulnerable option is the same as for the corresponding option without counterparty risk. This is akin to assuming that the option holder is myopic with respect to the possibility of default. In our second formulation, we suppose that the exercise strategy is not affected by the possibility of counterparty default. Therefore, we first compute the exercise barrier of the risk-free option at all exercise dates using the DP algorithm (4)-(5). The exercise barrier at t_m is denoted b_m and is obtained by finding the root of the equation

$$h_m(b_m) = [K-b_m]^+\,. \tag{8}$$

Denote by $v_m^C(s)$ the value of the vulnerable option at t_m when the price of the underlying asset is s under the myopic investor assumption. The holding value, adjusted for the possibility of default between dates t_m and t_{m+1} is then

$$h_m^C(s) = \beta \mathbb{E}_{m,s}[v_{m+1}^C\left(S_{t_{m+1}}\right)] - C_m(s)$$

where the adjustement is given by

$$C_m(s) = (1-R)\,\beta \mathbb{E}_{m,s}[1_{\{t_m<\tau\leq t_{m+1}\}} v_{m+1}^C\left(S_{t_{m+1}}\right)].$$

Assuming independence between default and asset price in our intensity model, we obtain similarly as above

$$\begin{aligned} C_m(s) &= (1-R)\left(1-e^{-\lambda\Delta}\right)\beta \mathbb{E}_{m,s}[v_{m+1}^C\left(S_{t_{m+1}}\right)] \\ h_m^C(s) &= \beta' \mathbb{E}_{m,s}[v_{m+1}^C\left(S_{t_{m+1}}\right)]. \end{aligned}$$

The only difference with respect to the previous model appears in the DP algorithm to compute the value of the vulnerable option, which becomes

$$\begin{aligned} v_m^C(s) &= \begin{cases} h_m^C(s) & \text{for } m=0 \\ \begin{cases} [K-s]^+ & \text{if } s \leq b_m \\ h_m^C(s) & \text{if } s > b_m \end{cases} & \text{for } m = 1,...,M-1 \\ [K-s]^+ & \text{for } m = M \end{cases} \\ h_m^C(s) &= \beta' \mathbb{E}_{m,s}[v_{m+1}^C(S_{t_{m+1}})], \quad \text{for } m = 0,...,M-1\,. \end{aligned}$$

Again, the CVA at t_m and s is given by

$$CVA_m(s) = v_m(s) - v_m^C(s).$$

The function $v_m^C(s)$ is discontinuous at b_m since $h_m^C(b_m) \neq h(b_m) = [K-b_m]^+$ so that care should be taken when interpolating it from discrete values. Notice that the value function equals the exercise payoff for $s \leq b_m$. In our implementation, we localize the grid points so that $a_i \in [b_m, \infty)$, $i = 0,...,p$ at step m, and the value function v_m^C is interpolated in the region where it is smooth.

Finally, we want to point out that, while the myopic assumption is used in practice, the fact that the value function is discontinous in this second model is a serious flaw, indicating arbitrage opportunities around the exercise barrier.

4 Numerical experiments

In this section, we report on numerical experiments comparing the precision and computational burden of our method with those of Monte-Carlo simulation, and illustrating the impact of WWR and of jumps in the asset price dynamics. All options are Bermudian puts with a maturity of one year and 100 exercise opportunities, which are good approximations of their American counterparts.

4.1 Simulation setup

Our simulation setup follows the first approach described in Section 2.3, that is, it assumes that the value and exercise strategy of the counterparty-risk-free option have already been computed and are available for all exercise dates as a function of the asset price. In our implementation, we use the function $\hat{v}_m(s)$ computed using the DP algorithm (6)-(7) and the exercise barrier b_m satisfying (8). Notice that, for $s > b_m$, $\hat{v}_m$ is a polynomial of degree p, completely characterized by its $p+1$ coefficients, whereas it is equal to the exercise payoff for $s \leq b_m$.

We then simulate the default time τ and the underlying asset price trajectory $\{S_{t_m}\}$, $m = 1, ..., M$ under their physical probability distributions, using antithetic variates to reduce simulation variance. For each sample path, we record the default time τ, the corresponding time index $j = \left\lceil \frac{\tau}{\Delta} \right\rceil$ such that $\tau \in (t_{j-1}, t_j]$, and the first date t_k at which the price of the underlying asset is below the exercise barrier b_k.

On a given sample path, if $j > k$, default occurs after the exercise of the option and the exposure is 0. If however $j \leq k$, default occurs during the time interval $(t_{j-1}, t_j]$ while the option is still alive, and the exposure is $(1 - R)\, \beta^j \hat{v}_j(S_{t_j})$. The CVA of the vulnerable option is obtained by averaging the exposures on all sample paths.

4.2 Lognormal model without WWR

Our first set of results reports on the computation of the CVA assuming a geometric Brownian motion model for the asset price dynamics, and assuming independence of default and market factors. The lognormal model and its parameters are recalled in Appendix 6.1. Tables 1 to 3 present results obtained assuming the option holder is myopic in his determination of the

p	50	100	150
CPU time	0.468	1.154	2.075

Table 1: CPU time according to the grid size, log-normal model without WWR.

exercise barrier, as described in Section 3.3. Results under the farsightedness assumption are qualitatively similar, but are obtained in half of the computation time. Simulation is performed using a sample of 1 000 000 scenarios, requiring around 600 CPU[1] seconds. We ran the DP algorithm with $p = 50$, 100 and 150 grid points and report on the corresponding CPU times in Table 1. All experiments were made using an Intel Core 2 Duo T6400 processor with 2 Ghz and 3 Go of RAM.

Tables 2 and 3 compare adjusted prices obtained using DP (prices for all grid sizes agreed up to the fifth decimal) to the 95% confidence intervals obtained by simulation for various option parameters and default intensity. The length of these confidence intervals is of the order of 10^{-3}; all DP prices are inside the intervals. One can observe the efficiency of our proposed approach in precision, computation time and memory: while 600 seconds are required to reach a precision of 10^{-3} by simulation using 10^6 samples, the DP approach reaches a precision of 10^{-5} in less than 0.5 seconds using 50 grid points.

Figure 2 compares the default-free and the adjusted value functions at inception. The difference between the two functions is the CVA, and is plotted in Figures 3 and 4 under both the farsighted and the myopic assumptions. One can observe that the CVA is important when the asset price is near the strike K. In fact, when the option in the money, counterparty risk is not very significant since the investor will generally exercise early, and it increases almost linearly with exposure. On the other hand, when the option is deep out of the money, the option value becomes so small that the adjustment eventually vanishes. Figure 4 shows the value of the CVA as a percentage of the value of the option. It is always the case that the CVA is slightly higher when the option holder is myopic than when he is farsighted.

Figure 5 compares the CVA of an at-the-money American put option to that of its European counterpart, as a function of the default intensity λ in

[1]Central Processing Unit (CPU) time is the amount of time used to process the instructions of the computer program.

		DP	**Simulation**
$S_0 = 47$	$\lambda = 0.05$	4.4001	$[4.3991, 4.4016]$
	$\lambda = 0.10$	4.3060	$[4.3043, 4.3077]$
	$\lambda = 0.15$	4.2146	$[4.2129, 4.2171]$
$S_0 = 50$	$\lambda = 0.05$	2.9594	$[2.9582, 2.9602]$
	$\lambda = 0.10$	2.8792	$[2.8782, 2.8811]$
	$\lambda = 0.15$	2.8017	$[2.8002, 2.8036]$
$S_0 = 53$	$\lambda = 0.05$	1.9383	$[1.9377, 1.9393]$
	$\lambda = 0.10$	1.8777	$[1.8765, 1.8788]$
	$\lambda = 0.15$	1.8191	$[1.8178, 1.8205]$

Table 2: Adjusted price of a Bermudian put option in the lognormal model without correlation. Parameters are $K = 50$, $r = 0.05$, $s = 0.2$, $T = 1$, $M = 100$.

		DP	**Simulation**
$S_0 = 56$	$\lambda = 0.1$	6.1803	$[6.1782, 6.1833]$
	$\lambda = 0.3$	5.6576	$[5.6521, 5.6603]$
	$\lambda = 0.5$	5.1918	$[5.1856, 5.1955]$
$S_0 = 60$	$\lambda = 0.1$	4.3424	$[4.3412, 4.3455]$
	$\lambda = 0.3$	3.8942	$[3.8929, 3.8998]$
	$\lambda = 0.5$	3.5002	$[3.4976, 3.5059]$
$S_0 = 64$	$\lambda = 0.1$	3.0049	$[3.0034, 3.0069]$
	$\lambda = 0.3$	2.6528	$[2.6488, 2.6545]$
	$\lambda = 0.5$	2.3467	$[2.3426, 2.3494]$

Table 3: Adjusted price of a Bermudian put option in the lognormal model without correlation. Parameters are $K = 60$, $r = 0.06$, $s = 0.25$, $T = 1$, $M = 100$.

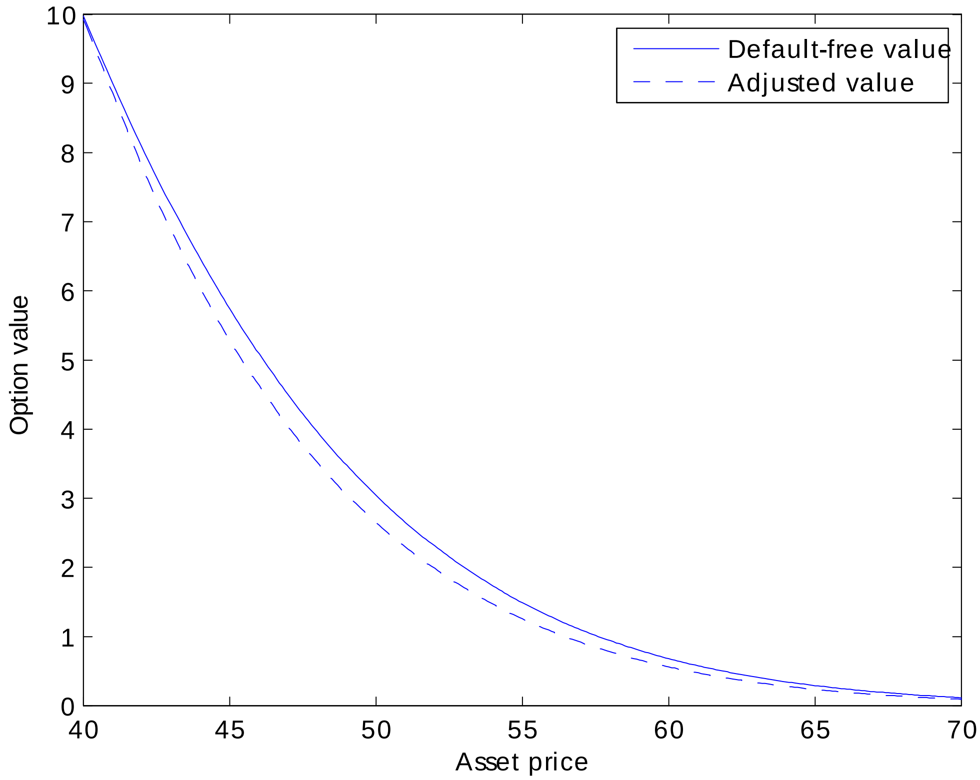

Figure 2: Comparison of the standard (default-free) and adjusted value of a Bermudian option in the lognormal model without correlation. Parameters are $K = 50,\ r = 0.05,\ \sigma = 0.2,\ T = 1,\ M = 100,\ \lambda = 0.25$.

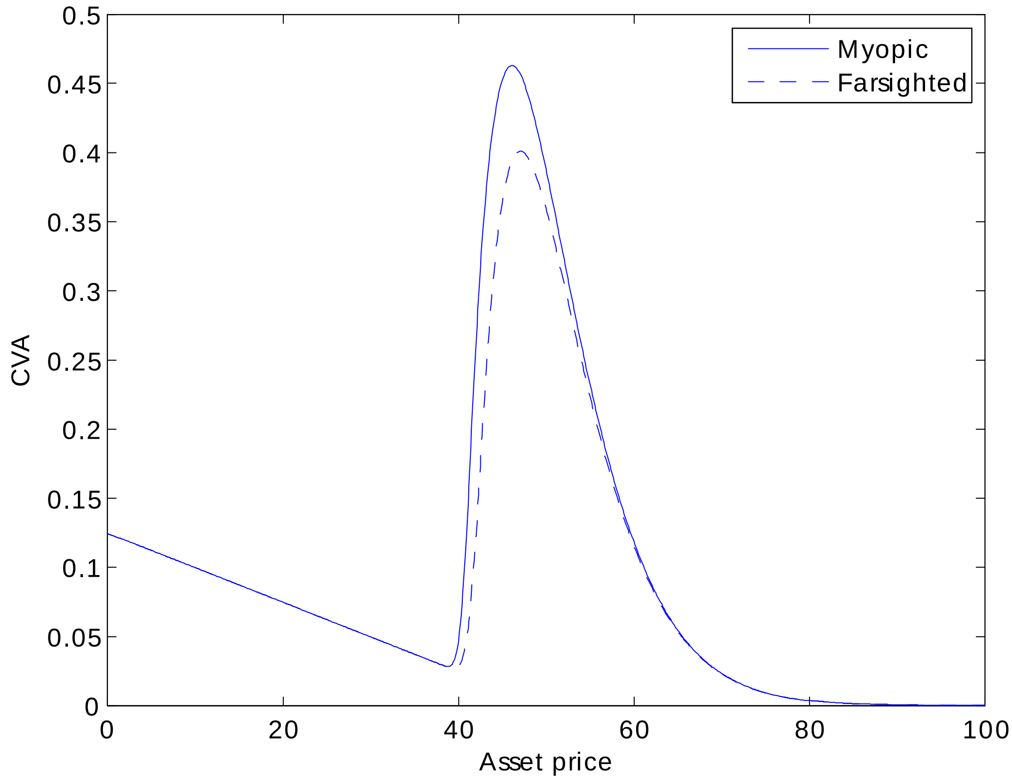

Figure 3: CVA of a Bermudian option in the lognormal model without correlation. Comparison between farsighted and myopic assumptions. Parameters are $K = 50,\ r = 0.05,\ \sigma = 0.2,\ T = 1,\ M = 100,\ \lambda = 0.25.$

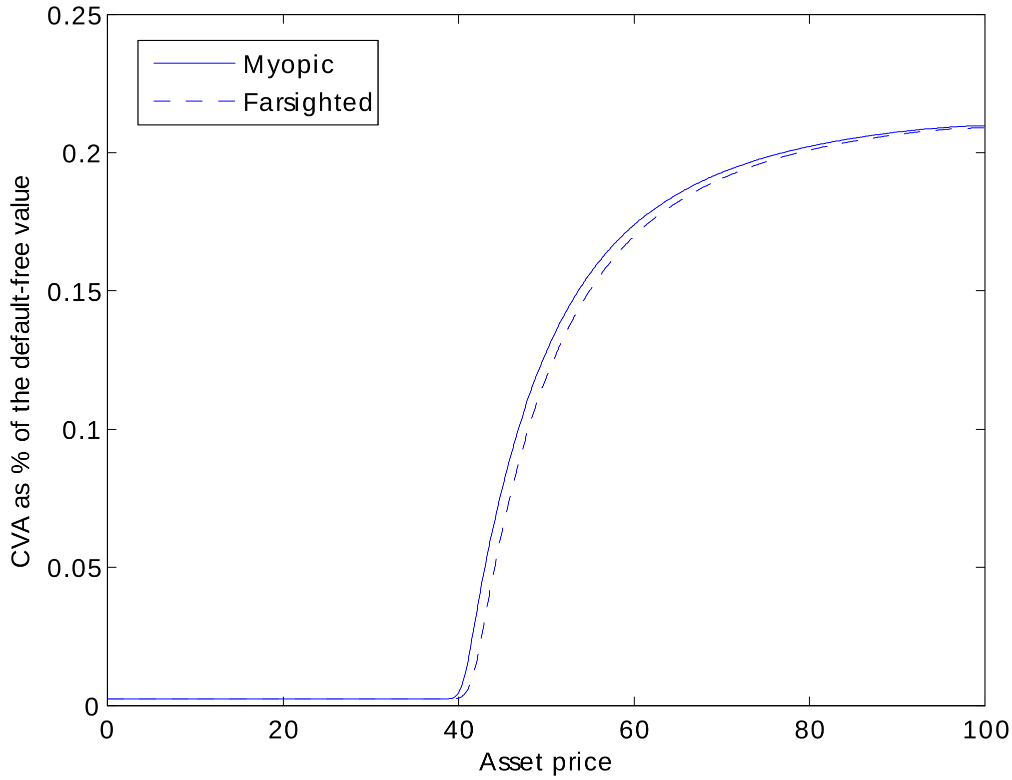

Figure 4: CVA as a proportion of the default-free value. Comparison between farsighted and myopic assumptions. Parameters are $K = 50$, $r = 0.05$, $\sigma = 0.2$, $T = 1$, $M = 100$, $\lambda = 0.25$.

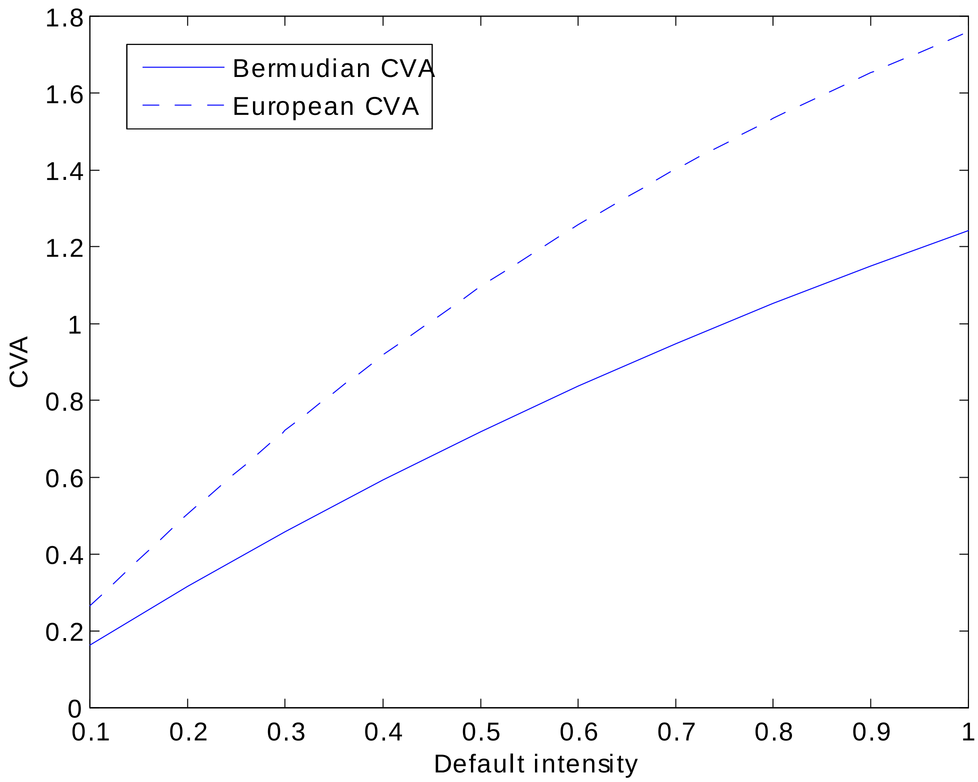

Figure 5: Comparison of the CVA of a European and a Bermudian option in the lognormal model without correlation. Parameters are $S_0 = 50,\ K = 50,\ r = 0.05,\ \sigma = 0.2,\ T = 1,\ M = 100.$

the myopic case. As expected, the CVA of an option is smaller with early exercise opportunities, because these reduce the exposure.

4.3 Lognormal model with WWR

Our second set of results reports on the impact of WWR on the CVA of a put option. We again assume that the asset price dynamics is described by a geometric Brownian motion. We experiment with two models proposed in the literature for the correlation between the default time and the asset price; the first model is based on a Gaussian copula as in Gregory (2010), and the second model uses a stochastic hazard rate related to the asset price, as in Hull & White (2012). Clearly, the introduction of WWR does not modify significantly the performance of the simulation approach: it suffices to account for the correlation model when generating the default time and asset price sample paths. This is also the case for DP: the inclusion of WWR modifies the conditional density functions of asset prices (in the first model) or the periodic discount rate (in the second model), which does not change the algorithm's complexity. Therefore, we present here qualitative results pertaining to the impact of WWR according to the two correlation models.

In the first model, correlation is measured by the parameter ρ representing the correlation between the Brownian component of the price and a transformation of the default time. Figure 6 compares the CVA at inception (under the myopic assumption) of a Bermudian put option with a large number of exercise possibilities to that of its corresponding European counterpart, as a function of the correlation coefficient in this model. One can observe that, with respect to European options, the impact of WWR when the option holder has early exercise rights is very small: the increase in the CVA when correlation increases is barely observable in the Bermudian case. This agrees with what is observed in the literature: under WWR, counterparty risk does not reduce much the value of American-style instruments because of the possibility of exiting the contract given to the option holder. Figure 7 plots the part of the CVA that is due to WWR for a correlation of 0.75 in the Gregory (2010) model, as a function of the asset price .

In the second model, the dependence between the intensity of default and the asset value is described by two parameters denoted by c and d, where d measures the amount of Right Way or Wrong Way risk ($d > 0$ indicates WWR). Accordingly, the default intensity is assumed to be $\lambda_{ts} = \exp(c + dv_m(s))$ during the time interval $[t_m, t_{m+1}]$ when $S_{t_m} = s$. Figure 8 illustrates the difference in the CVA of a vulnerable Bermudian option with or without WWR, where parameters c and d are set so that $\lambda_{ts} \in [0.25, 1]$, and where the CVA without WWR is computed with $\lambda = 0.25$.

The two models seem to behave similarly, except for deep in the money options, which most probably would not be available at inception. Figures 9 and 10 compare the additional CVA of the two models (in the myopic framework) at the first exercise date. When the option holder can exercise his option, the two models behave in a similar way; in both cases, WWR is highest around the exercise barrier.

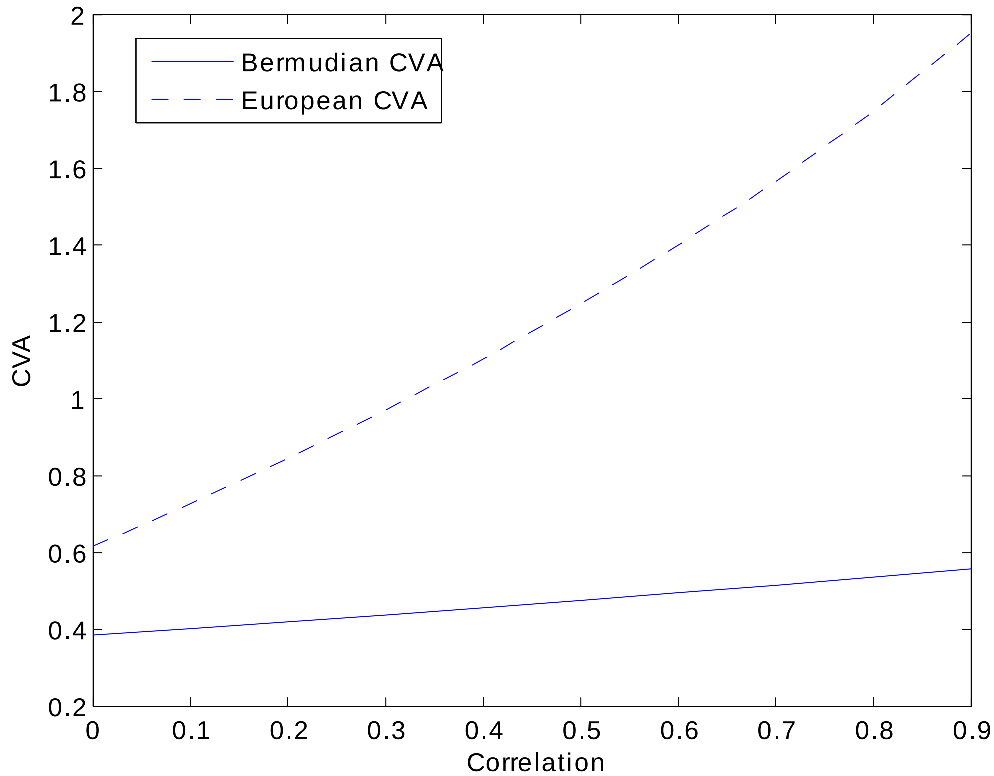

Figure 6: CVA of Bermudian and European options as a function of the correlation ρ in the Gregory (2010) model. Parameters are $S_0 = 50,\ K = 50,\ r = 0.05,\ \sigma = 0.2,\ T = 1,\ M = 100,\ \lambda = 0.25$.

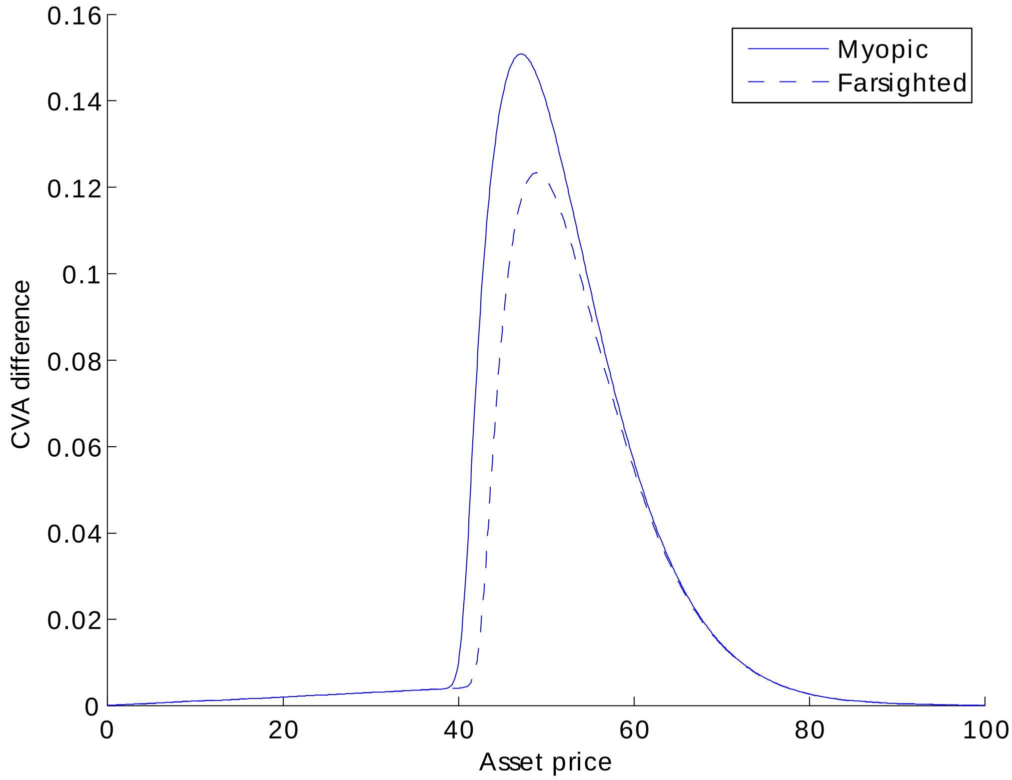

Figure 7: Additional CVA due to WWR in the Gregory (2010) model. Parameters are $K = 50$, $r = 0.05$, $T = 1$, $M = 100$, $\sigma = 0.2$, $\lambda = 0.25$, $\rho = 0.75$.

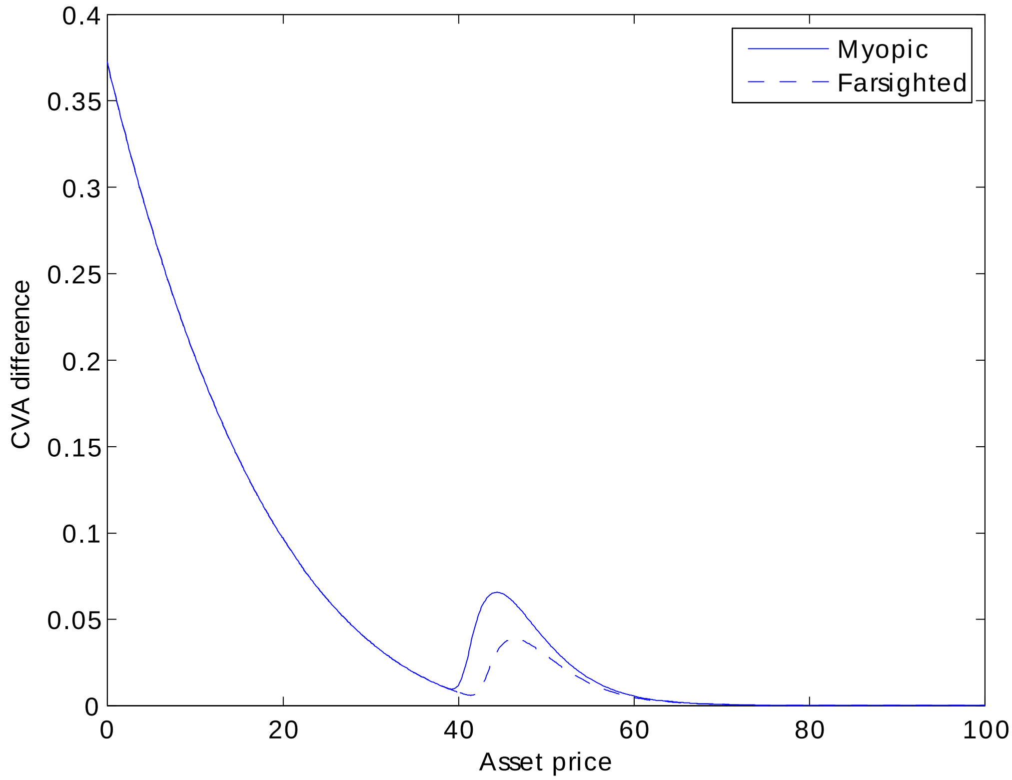

Figure 8: Additional CVA due to WWR in the Hull & White (2012) model. Parameters are $K = 50$, $r = 0.05$, $T = 1$, $M = 100$, $\sigma = 0.2$, $\lambda = 0.25$, $c = -1.3863$, $d = 0.0277$.

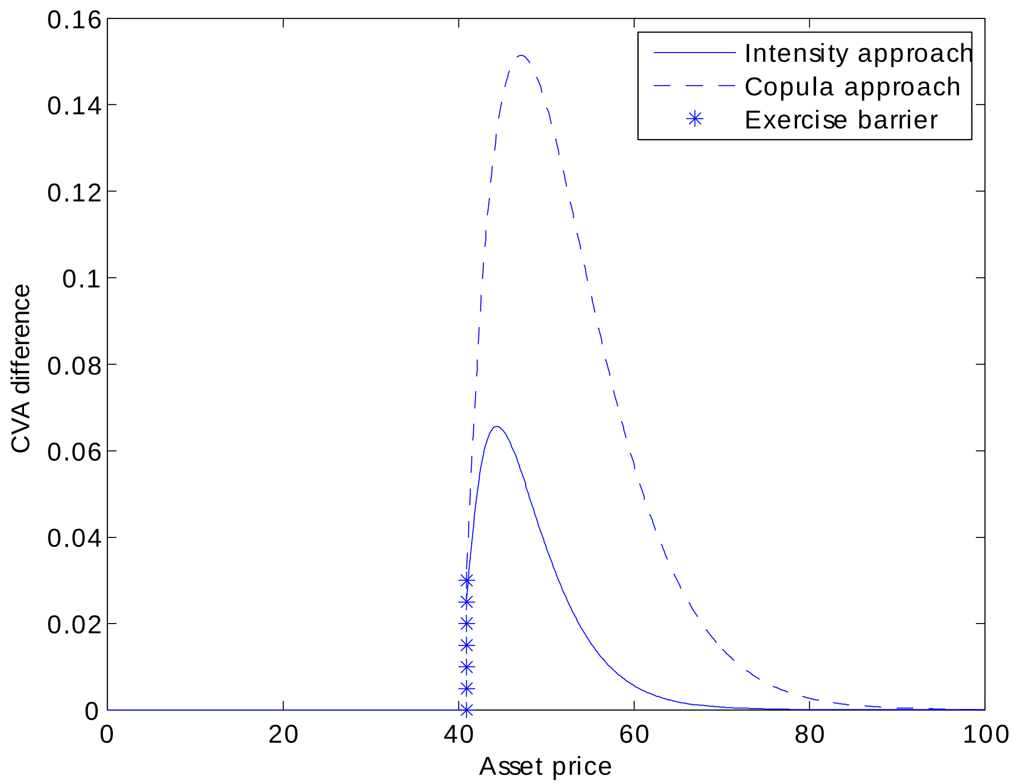

Figure 9: Additional CVA due to WWR (myopic assumption) at the first exercise date. $K = 50$, $r = 0.05$, $T = 1$, $M = 100$, $\sigma = 0.2$, $\lambda = 0.25$, $\rho = 0.75$ (for the copula approach), $c = -1.3863$, $d = 0.0277$ (for the intensity approach).

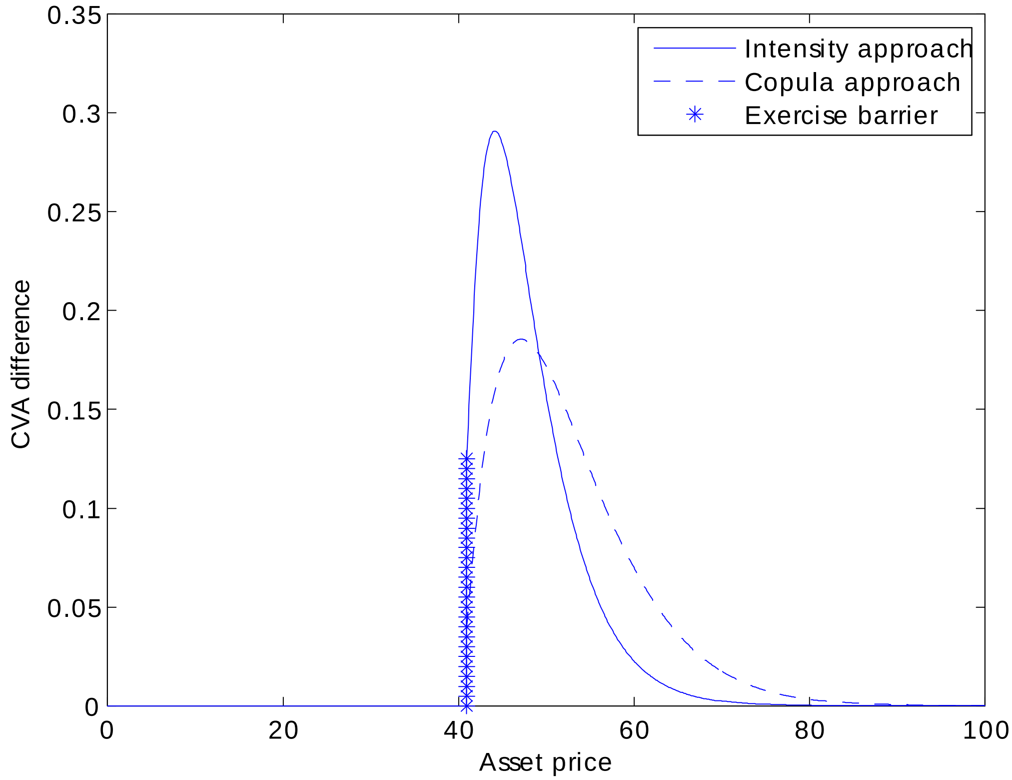

Figure 10: Additional CVA due to WWR (myopic assumption) at the first exercise date. $K = 50$, $r = 0.05$, $T = 1$, $M = 100$, $\sigma = 0.2$, $\lambda = 0.25$, $\rho = 0.9$ (for the copula approach), $c = -1.3863$, $d = 0.1$ (for the intensity approach).

p	60	80	100	120
λ =0	8.369	8.346	8.346	8.346
λ =0.25	7.077	7.058	7.058	7.058
λ =0.50	6.008	5.993	5.993	5.993
λ =1	4.383	4.373	4.373	4.373
λ =2	2.453	2.448	2.448	2.448
CPU time	5.59	7.74	10.12	12.68

Table 4: Price of vulnerable Bermudian puts. Parameters are $S_0 = 50$, $K = 50$, $r = 0.05$, $\sigma = 0.2$, $\alpha = 5$, $\gamma = 0$, $\psi = 0.2$, $T = 1$, $M = 50$.

4.4 Jump-diffusion model

Our third set of results is obtained by specifying a different market model, namely the Jump-diffusion model of Merton (1976). In turbulent times when counterparty risk is present, jumps in the asset prices may well be a better assumption than the constant volatility of the lognormal model. We investigate in this section to what extent the introduction of the possibility of jumps in the asset price's dynamics is significant when dealing with counterparty risk. Our results are obtained under the assumptions that asset prices and default time are independent and that the exercise strategy is myopic.

The jump-diffusion model and parameters are recalled in Appendix 6.2. As pointed out in Section 3.1, the DP algorithm can accomodate any market model. However, a more complex model and added volatility may require additional processing time and number of grid points to attain a given precision. For a risk-free option, we reach an accuracy of 10^{-4} with respect to the analytical solution of Merton (European option) and the compound option approach of Gukhal (2004) (two exercise possibilities) with 10 grid points in 0.03 seconds. With 50 exercise possibilities, convergence to four digit precision is attained in 4 seconds with 80 grid points.

Table 4 presents the prices of vulnerable Bermudian options obtained by our DP approach for various grid sizes, as a function of the default intensity. It shows that a precision of 10^{-3} is attained with 80 grid points in around 8 seconds.

Figure 11 presents the impact of the presence of jumps on counterparty risk for a Bermudian put option. It compares the CVA in the jump-diffusion framework to the CVA in the pure diffusion model, as a function of the default intensity. One observes that the introduction of jump risk in the model has a

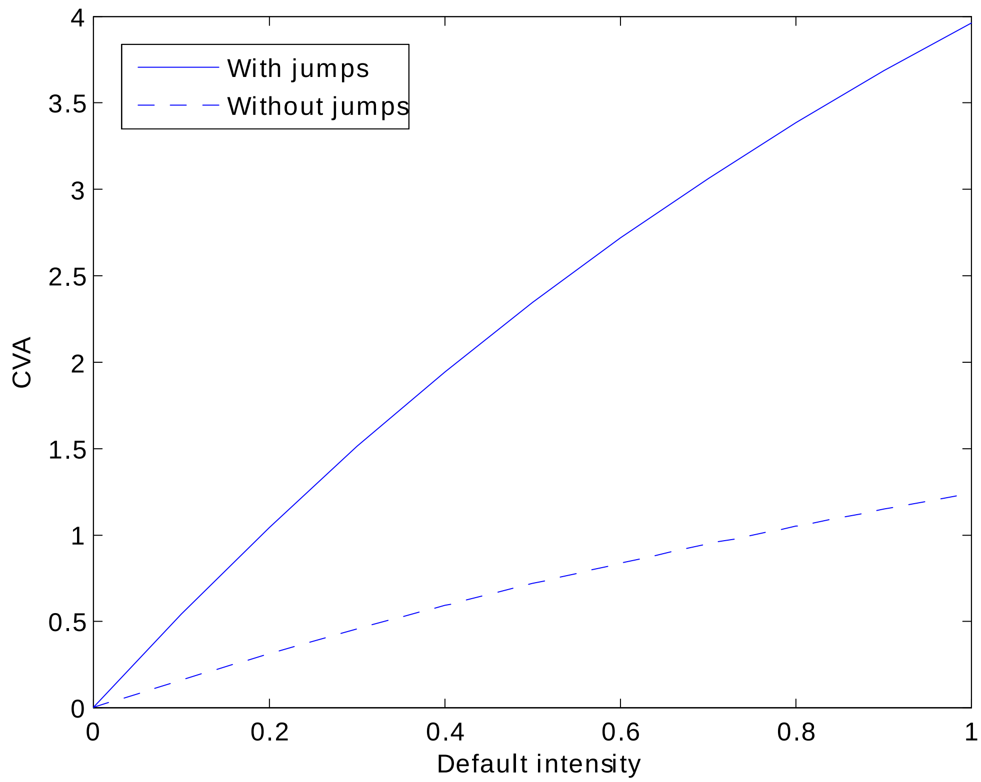

Figure 11: CVA with jumps ($\alpha = 5$) and without jumps ($\alpha = 0$) as a function of λ. Parameters are $S_0 = 50,\ K = 50,\ r = 0.05,\ \sigma = 0.2,\ \gamma = 0,\ \psi = 0.2,$ $T = 1,\ M = 100.$

significant impact on the CVA, which is increasing with the default intensity.

5 Conclusion

We propose a new efficient DP approach to price options with early exercise opportunities subject to counterparty risk. Computation of the CVA then reduces to the computation of the difference between the value of default-free and vulnerable options with the same characteristics. The method is very general and highly flexible. With respect to simulation-based methods, our approach yields the CVA with better precision, using much less computation time and memory. Moreover, unlike simulation, it provides the CVA for all possible values for the underlying asset's price and time to maturity.

We provide numerical illustrations for the classic lognormal and jump-diffusion models, and for two models for WWR, using Bermudian put options with a large number of exercise opportunities. The DP method can accomodate any market model, provided density functions can be evaluated efficiently, many counterparty risk features, such as WWR and the presence of collateral, and most derivative contracts, including exotic and path dependent options. However, like all DP approaches, since the value function is evaluated for all possible states of the world, our method is best suited for market models and contracts with low-dimensional factors.

6 Appendix

6.1 Lognormal model

In the lognormal model, the price process S_t is described by:

$$dS_t = \sigma S_t dW_t + \mu S_t dt,$$

where:

- W_t is a Brownian motion

- σ is the volatility

- μ is the drift.

Under the risk-neutral measure, the price dynamics is then described by:

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{t}X\right),$$

where:

- X is a standard Gaussian random variable

- r is the risk-free rate.

6.2 Merton jump-diffusion model

In the jump-diffusion model, the price process is described by

$$dS_t = (\mu - y\alpha)S_t \; dt + \sigma S_t \; dW_t + S \; dQ_t \,, \tag{9}$$

where

- μ is the drift
- α is the intensity of the process Q_t
- y is the expected jump size
- σ is the volatility
- W_t is a Brownian motion
- Q_t is a compound Poisson process.

Under the risk-neutral measure, the price dynamics is then described by:

$$S_t = S_0 \exp\left(\sigma X + (r - y\alpha - \frac{1}{2}\sigma^2)t\right) \prod_{i=1}^{N_t} (Y_i + 1).$$

where:

- X is a standard Gaussian random variable
- Y is a random variable
- r is the risk-free rate.

In the Merton model, the distribution of the jump size Y is log-normal:

$$\log(Y + 1) \sim \mathcal{N}(\gamma, \psi^2).$$

where γ and ψ are respectively the mean and the standard deviation of the log jump sizes.

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